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OF THE

AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

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1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: 1, 2, $\cdots$, n.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: l. Similarly, 0 means "zero"; to indicate "cap O," backspace and overprint period; thus: O.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma\ \cdots$ (all) A B $\Gamma\ \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma\ \cdots}$ (all except o).

★ Light Greek Inferiors—$_{\Delta\Lambda\Sigma\Omega}$ and $_{\alpha\ \beta\ \gamma\ \cdots}$ (all except o).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \gamma\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \kappa\ \lambda\ \mu\ \nu\ \xi\ \pi\ \rho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ f\ p\ q}$ $\mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ K\ L\ M\ P\ Q\ R}$ $\mathfrak{S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{b}$ $\mathfrak{A\ B\ D}$.

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* Hebrew— ℵ א ב ב troublesome to handle.

★ Dashed Italics—$\overline{A}\ \overline{a}\ \overline{B}\ \overline{b}\ \overline{C}\ \overline{c}\ \overline{E}\ \overline{e}\ \overline{F}\ \overline{f}\ \overline{G}\ \overline{g}\ \overline{H}\ \overline{I}\ \overline{\imath}\ \overline{j}\ \overline{K}\ \overline{k}\ \overline{M}\ \overline{m}\ \overline{N}\ \overline{n}\ \overline{o}\ \overline{P}\ \overline{p}$ $\overline{q}\ \overline{R}\ \overline{r}\ \overline{S}\ \overline{t}\ \overline{u}\ \overline{v}\ \overline{X}\ \overline{x}\ \overline{Y}\ \overline{y}\ \overline{Z}\ \overline{z}$.

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$.

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\varepsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$.

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \bar{\Gamma}$.

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \dot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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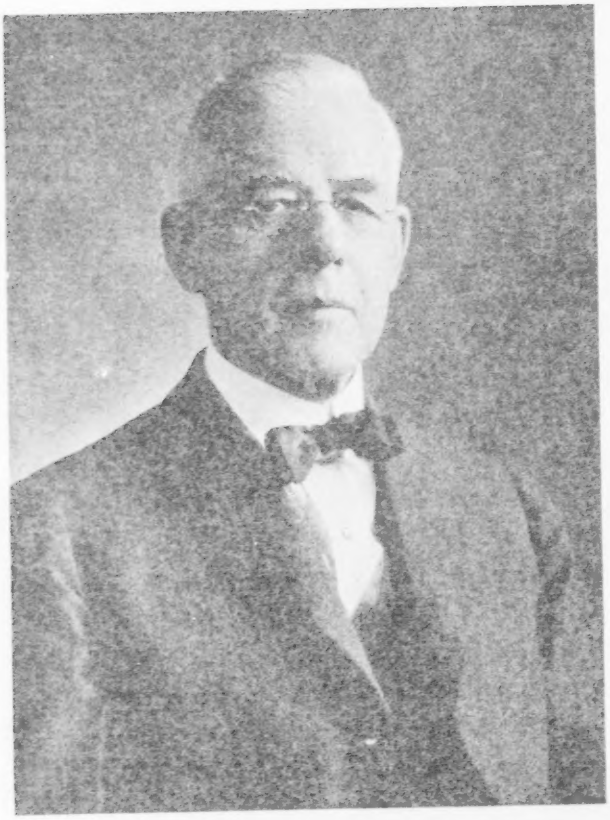

HENRY BURCHARD FINE
1858–1928
President of the American Mathematical Society
1911–1912

THE JUNE MEETING IN BERKELEY

The two hundred seventieth regular meeting of the Society was held at the University of California on Thursday and Friday, June 20–21, 1929, in conjunction with the June 19–22 meetings of the American Association for the Advancement of Science. Sessions were held Thursday morning, Thursday afternoon, and Friday morning. The attendance included the following thirty-three members:

Alderton, J. P. Ballantine, Bateman, Bernstein, Biggerstaff, Cajori, Coleman, Corbin, Daus, Doole, G. C. Edwards, Garver, J. L. Gibson, W. L. Hart, M. W. Haskell, E. R. Hedrick, Hicks, Jerbert, D. N. Lehmer, Lenzen, Leonard, S. H. Levy, McFarlan, F. R. Morris, Pierpont, T. M. Putnam, J. M. Rankin, Pauline Sperry, Morgan Ward, G. T. Whyburn, W. M. Whyburn, A. R. Williams, B. C. Wong.

There was no meeting of the Council or of the Trustees of the Society.

The meeting was presided over at various times by President Hedrick and Professors M. W. Haskell, W. L. Hart James Pierpont, and Harry Bateman.

A luncheon for members and their guests was held at the Women's Faculty Club between the Thursday sessions.

At the Thursday morning session, by invitation of the Program Committee, Professor James Pierpont, of Yale University, delivered an address entitled *Non-euclidean geometry in retrospect*. The address will appear in full in a later issue of this Bulletin.

Titles and abstracts of other papers read at the meeting follow. The papers of the following authors were read by title: Clifford Bell, E. T. Bell, Brown, Cook, Craig, Hollcroft, Jones, Milne, Pierce, Roberts, Robertson, Rutt, Weyl, G. T. Whyburn (second paper), W. M. Whyburn (third paper), Wilder, Woodard, Zippin.

1. Professor J. P. Ballantine: *A vector derivation of Cramer's rule.*

A system of two equations in two unknowns may be written $\alpha x+\beta y=\gamma$, where $\alpha=(a_1, a_2)$ and similarly β, γ are vectors. It is easily seen from a diagram that x is the ratio between the altitudes on the common side β, and hence between the areas of the parallelograms $\gamma\beta$ and $\alpha\beta$. Hence the determinantal formula. The procedure generalizes readily to any number of dimensions. The paper will appear in the American Mathematical Monthly.

2. Professor Harry Bateman: *Some properties of spheroidal harmonics.*

The harmonics of even order associated with an oblate spheroid are expressed as definite integrals involving the corresponding harmonics for a prolate spheroid and some other interesting relations are obtained.

3. Dr. Clifford Bell: *Polygons jointly related to the rational cubics and other curves.*

A study is made of the n-sided polygon having $n-1$ of its vertices on, and all its sides tangent to, a rational cubic, the remaining vertex lying on any one of a finite set of algebraic curves $f_i(x, y, z)=0$ of orders m_i. The number of such n-sided polygons so connected with the cuspidal cubic is $3\sum_{i=1}^{s} m_i$ where s is the total number of curves $f_i(x, y, z)=0$. The number for the nodal cases is given by $3\cdot 2^{n-1}\sum_{i=1}^{s} m_i$. The vertices of the polygons are ordered, and in the cuspidal case it is shown that by a simple magnification of x, y, and z, dependent only on n and the affix of the vertices, any one of the curves of the set $f_i(x, y, z)=0$ can be deformed in such a way as to contain all of the vertices of given affix belonging to polygons associated with this curve and the given cubic. The well known results for triangles in and circumscribed to the rational cubics are obtained by specializing to the case $n=3$ and allowing the curves $f_i(x, y, z)=0$ to become coincident with the cubic.

4. Professor E. T. Bell: *Theorems on total representations as sums of squares or triangular numbers.*

If m, n are coprime integers >0, and $f(x)$ is uniform and finite for integer values of x, $f(x)$ is said to be factorable if $f(mn)=f(m)f(n)$. Denote by $O(n), E(n)$ the total numbers of representations of n as sums of an odd, an even number of squares of integers $\gtrless 0$. Neither of $O(n)$, $E(n)$ is factorable. In a former note (American Mathematical Monthly, vol. 30 (1923), p. 441), it was shown that $O(2n+1)>E(2n+1)$, $E(2n)>O(2n)$ for all $n>0$. Here this property is widely generalized. As a very special case, $(-1)^{n-1}[\sum(-1)^j Q_j(n)/j]$ where $Q_j(n)=$twice the number of representations of n as a sum of j squares with integer roots $\neq 0$, is factorable, where $\sum$ refers to $j=1, 2, \cdots, n$; the sums of the positive, negative terms respectively satisfy the same inequalities as E, O. Further, the number of representations may be replaced by any even function of the actual integers in the respective representations, and the corresponding theorems then stated. The like is carried out for representations as sums of positive

triangular numbers. The results are scarcely obvious by purely arithmetical methods; by the means used in the proofs given here, the theorems may be extended indefinitely.

5. Professor E. T. Bell: *Non-existence theorems on the number of representations of arbitrary odd integers as sums of $4r$ squares.*

This paper appears in full in the present number of this Bulletin.

6. Professor E. T. Bell: *Existence theorems on the number of representations of odd integers as sums of $4t+2$ squares.*

The number $N(n, 2r)$ of representations of n as a sum of $2r$ squares, consists of two parts: first, a polynomial in the real divisors of n; second, a function which, for n general, is not reducible to a polynomial in the real divisors of n. But if n be restricted to be a member of certain arithmetic progressions, the second part may vanish identically. In the preceding paper, the necessary and sufficient conditions that this be so for $N(m, 4r)$, m odd, are determined. A remarkable transcendental equation in r is the decisive condition. This paper closes, negatively, a reasonable conjecture arising from the last section of Eisenstein's classic paper, Crelle's Journal, vol. 34 (1847), p. 35. In order that $N(m, 4r+2)$, $m \equiv 3 \bmod 4$, $r>2$, shall be a polynomial in the real divisors of m, it is necessary (but not sufficient) that

$$105(7^{2r}-1) = (512r^4-1280r^3+1120r^2+440r-162)(3^{2r}-1).$$

A similar condition holds for $m \equiv 1 \bmod 4$. Necessary and sufficient conditions, much more drastic, also are obtained, and thence the "divisor" character of $N(m, 4r+2)$ is settled.

7. Professor E. T. Bell: *On the representation of $4t+2$ as a sum of $4s$ squares.*

See the preceding abstract.

8. Professor E. T. Bell: *On the application of elliptic functions to universal and quasi-universal forms.*

A form f of degree 2 that represents all integers >0, or all integers $\geqq 0$, is called (after Dickson), universal. If f represents all integers of a given type, that type to be finitely definable, f is here called quasi-universal where f is not universal. By the method of paraphrase much information concerning universal or quasi-universal forms can be obtained from extremely simple analysis. By the same method an immediate extension to forms of degree >2 is attainable. The possibilities of this application of elliptic functions being unlimited, only a selection is given, sufficient to show the method.

9. Professor B. A. Bernstein: *On Whitehead and Russell's theory of deduction.*

Whitehead and Russell in their *Principia Mathematica* develop a theory of deduction as basis for the entire structure of the *Principia*. The present author discusses the nature of this theory and its relation to Boolean algebra.

10. Professor B. A. Bernstein: *Application of Boolean algebra to proving consistency and independence of postulates.*

The author shows how the problem of determining the consistency or the independence of a set of postulates can often be reduced to the solvable problem of determining the simultaneity of a set of Boolean relations.

11. Mr. A. B. Brown: *Relations between the critical points of a real analytic function of n independent variables.*

Morse's paper *Relations between the critical points of a real function of n independent variables* (Transactions of this Society, vol. 27, 1925) treats the case of non-degenerate critical points, that is, where the Hessian of the function f is not zero at any critical point. In the present paper the author investigates the case in which this restriction is dropped, but requires f to be analytic and have isolated critical points. Given a critical point at which $f=c$, not a point of minimum or maximum, a small $(n-1)$-sphere with center at the critical point is taken, and a corresponding small constant $e>0$. The kth type number of the critical point, $k=1, 2, \cdots, n-1$, is defined as the $(k-1)$st order connectivity number, diminished by unity, of the complex of points of the locus $f=c-e$ within or on the sphere. Under this count Morse's relations are found to hold.

12. Professor Florian Cajori: *Generalizations in geometry as seen in the history of developable surfaces.*

During the period of primitive intuition in the study of surfaces, only cones and cylinders were known to be developable. There followed three stages of generalization: (1) by the application of infinitesimal analysis to real surfaces, the discovery of other real developable surfaces, as a special type of ruled surfaces; (2) by the freer use, in analysis, of complex numbers, the discovery of developable imaginary surfaces; (3) by transcending the limitations of infinitesimal analysis, the discovery of real developable surfaces which are not ruled.

13. Professor Florian Cajori: *New data on the origin and spread of the dollar mark.*

West Indian manuscripts of 1760 and 1778 link the \$ unmistakably with the Mexican sign p^s for pesos or piastres, the s being brought down upon the p. Nineteenth century manuscripts of Lower California furnish cumulative evidence in support of this origin. The symbol \$ spread from the West Indies and New Orleans to New York, the Mississippi region, and to other places in commercial relations with the West Indies and Gulf Coast.

14. Professor Florian Cajori: *Absurdities due to division by zero. A historical note.*

Bishop Berkeley declared division by zero "shocking to good sense." The first to prove rigorously that division by zero is inadmissible was Martin Ohm. The first openly to derive absurdities from division by zero was Bernard Bolzano. The first to list the derivation of such absurdities as a mathematical recreation was W. W. R. Ball.

15. Professor A. J. Cook: *Pairs of rectilinear congruences with generators in one-to-one correspondence.*

When corresponding generators of a pair of rectilinear congruences do not intersect, a general theory for the projective study of the configuration may be constructed from a system of linear partial homogeneous differental equations of the first order in four dependent and two independent variables. The theory seems to be well adapted for the study of certain correspondences of Fubini, and a complete basis is furnished for their further investigation. The relation of Fubini's work to the earlier work of Darboux is also fully established. In connection with these correspondences a generalization to surfaces is secured for the concept of intersector curve which Lane introduced in his study of pairs of ruled surfaces.

16. Mr. H. V. Craig: *On the simultaneous invariants of two functions.*

This paper contains an investigation of the simultaneous invariants of two functions $F(x,x',x'')$ and $H(x,x')$. The chief result is a theory of parallelism developed from a fundamental tensor whose components are functions of x,x', and x''. The magnitude of vectors and the cosine of the angle between vectors undergoing parallel displacement are constant.

17. Professor P. H. Daus: *Note on the solution of* $x^3+2=y^2$.

The solutions of (1) $x^3+2=y^2$ in integers can be obtained from those of (2) $\alpha^3+3\alpha\beta^2-2\beta^3=1$ and conversely. The fact that (2) has solutions implies that the units in the cubic domain defined by $\theta^3+3\theta-2=0$ have the form $x+\theta y+\theta^2 z$, with $x=X^3+3Y^3$, $y=XY$, $z=Y^2$. The units in the cubic domain are connected by the recursion formulas $s_n=3(x_1-2z_1)s_{n-1}-3(x_{-1}-2z_{-1})s_{n-2}+s_{n-3}$, where (x_1, y_1, z_1) is the fundamental unit and (x_{-1}, y_{-1}, z_{-1}) its reciprocal. Since $s_1=(1, -1, -1)$, these recursion formulas can be used to prove (2) has no solution other than the trivial one $\alpha=1$, $\beta=0$, and (1) has only the solution $x=-1$, $y=\pm 1$. Several proofs of this fact have recently been given by other methods. The method outlined here may be extended to other equations of the type $x^3+a=y^2$.

18. Mr. H. P. Doole: *Certain multiple-parameter expansions.*

This paper deals with the proof of the convergence of the expansion of an arbitrary function in terms of the solutions of certain multiple-parameter

systems of linear differential equations of the first order in two variables. In §2 the solutions of $X_i' + [\lambda a_{i1}(x_i) \pm \mu a_{i2}(x_i)]X_i = 0$, $(i = 1, 2)$, are used with the boundary conditions $X_i(a_i) = \gamma_i X_i(b_i)$, $\gamma_i \neq 0$. In §3 the system $X_i' + (\lambda x_i^{h_i} \pm \mu)X_i = 0$, $X_i(-a) = X_i(b)$, $(i = 1, 2)$, has particular interest in the fact that $x_i^{h_i}$ changes sign in the interval considered. Convergence is shown for the case where h_i is integral, and possible divergence when h_i is such that $x_i^{h_i}$ is complex. In §4 convergence is proved for a certain system which has one auxiliary condition at more than two points of the interval, namely: $X(-1) - 2\alpha X(0) - X(1) = 0$. In §§2–3, the number of parameters might have been extended to n.

19. Professor Raymond Garver: *On the Brioschi normal quintic.*

The most direct reduction of a quintic equation to the Brioschi normal form seems to be that given by Gordan in volume 28 of the Mathematische Annalen, and improved successively by Weber and Dickson. However, the transformation which effects the reduction is valid only when a rather large number of restrictions is imposed on the given quintic. The present paper considers the significance of these restrictions (the matter has apparently not been considered before), and supplies valid transformations for two types of quintics to which the Gordan transformation cannot be applied.

20. Professor W. L. Hart: *Limited trilinear forms in Hilbert space.*

If a point $x(x_1, x_2, \cdots)$ lies in real Hilbert space, then $(\sum_1 x_i^2)^{1/2}$ will be called the modulus of x. The author considers real trilinear forms $A(x, y, z) = \sum_{i,j,k=1}^{\infty} a_{ijk} x_i y_j z_k$ where (x, y, z) are in H-space. By definition, A is limited if, for some $K > 0$, $\left| \sum_{i,j,k=1}^{n} a_{ijk} x_i y_j z_k \right| \leqq K$ for all n and for all (x, y, z) whose moduli are at most 1. It is proved that if, for any specified order of summation, $A(x, y, z)$ converges for all (x, y, z) in H-space, then $A(x, y, z)$ is uniformly bounded for all (x, y, z) whose moduli are at most 1, and hence A is limited. The method used in proving this theorem would apply as well to the proof of the uniform finiteness property for multilinear forms of any multiplicity. By use of familiar methods, it is proved that, if A is limited, then its formal series converges either as a triple series or as an iterated sum, for each possible choice as to the order of summation among the indices (i, j, k). Demonstrations are also given for other easily anticipated properties of limited trilinear forms.

21. Professor M. W. Haskell: *The cylindroid in non-euclidean geometry.*

The cylindroid is the locus of the axes of a pencil of linear complexes. In euclidean geometry this is a surface of the third order and of the third class. In non-euclidean geometry, however, a linear complex will have a pair of axes, which are mutually polar with respect to the absolute, and the locus of the axes of a pencil is a surface of the fourth order and fourth class. We shall use rectangular coordinates, so that the equation of the absolute

is $x^2+y^2+z^2+k^2t^2=0$. If the special complexes of the pencil are real, their axes may be taken symmetrically as the lines

$$y-mx=0,\quad z-ct=0;\quad y+mx=0,\quad z+ct=0.$$

The equation of the cylindroid is then, in point-coordinates,

$$c(m^2+1)xy(z^2/k^2+t^2)=m(1+c^2/k^2)zt(x^2+y^2)$$

and, in plane coordinates,

$$c(m^2+1)\xi\eta(h^2+\theta^2/k^2)=m(1+c^2/k^2)h\theta(\xi^2+\eta^2),$$

which, as $k=\infty$, reduce to the well known equations of the euclidean geometry. The surface is invariant under four infinite sets of collineations and is also autopolar with respect to the four pencils of quadric surfaces

$$x^2+y^2+2\lambda zt=0,\quad xy+\lambda zt=0,\quad x^2+y^2+\lambda(z^2+k^2t^2)=0,\quad xy+\lambda(z^2+k^2t^2)=0.$$

The cylindroid can be shown to be the intersection of two linear complexes and of a quadratic complex.

22. Professor E. R. Hedrick: *Generalizations of Cauchy's integral theorem.*

In this paper, the author indicates generalizations of the well known Cauchy integral theorem for non-analytic functions of a complex variable.

23. Professor T. R. Hollcroft: *Multiple points of algebraic curves.*

In 1750, Cramer determined the maximum number of multiple points that may occur on algebraic curves of orders up to and including eight. In 1839, Plücker derived a general limit to the number of double points of an algebraic curve of given order. For multiple points of orders higher than two, limits have been found only in the case of individual curves. In the present paper, general limits are obtained defining the maximum number of multiple points of given order that may belong to an algebraic curve of given order. Two limits are found. When the order of the multiple point lies within a certain range of values, one of these limits applies; otherwise, the other.

24. Professor Glenn James: *Generalizations of Pascal's and Brianchon's theorems.*

Considering, in the case of Pascal's Theorem, that five vertices of a given hexagon are fixed, this paper studies the locus of the sixth vertex when the opposite sides of the hexagon are not necessarily collinear but various restrictions have been placed upon the triangle which they form. For example, if the determinant of the equations, (in cartesian coordinates), of the intersections of opposite sides of the hexagon is equal to a constant, the vertices of the hexagon describe twelve conics which reduce to the single conic on the vertices of the hexagon when the given constant is zero.

25. Dr. B. W. Jones: *On two regular positive quadratic ternary forms.*

Previous to the appearance of this paper there has been no proof, as far as the author knows, that the form $x^2+8y^2+32z^2$ represents all positive integers of the form $8n+1$ and that $x^2+2y^2+32z^2$ represents all positive integers of the forms $8n+1$ and $8n+3$. The Dirichlet method does not apply in this connection. Here, using some results derived by elliptic functions together with congruence theory, these proofs are given, thus completing the proof of the regularity of these two forms.

26. Professor L. H. McFarlan: *Sets of necessary and sufficient conditions for the most simple problem of the calculus of variations.*

Bolza in his *Vorlesungen über Variationsrechnung* establishes a set of five necessary conditions on an arc joining two fixed points of the plane, which minimizes an integral of the form $\int f(x, y, p)dx$, $p=dy/dx$. He also gives a set of sufficient conditions on such an arc in order that it should have the above property. These sets are not identical. In his treatment of the corresponding problem in the parametric form one set of conditions is given which is shown to be both necessary and sufficient in order that the arc have the minimizing property. In some respects the two problems treated may be shown to differ. For example the minimizing arc in the non-parametric form of the problem must not have a tangent parallel to the y axis. Such a restriction is removed in the parametric treatment of the problem. There exists however a class of problems which may be treated by either method. The existence of a set of necessary and sufficient conditions on the minimizing arc in the parametric problem indicates the existence of one such set in the other form. The problem is studied with the intention of closing the apparent gap existing between the set of necessary and the set of sufficient conditions in the non-parametric form of the problem by transforming the single set of conditions in the parametric problem into the non-parametric form.

27. Professor L. H. McFarlan: *Sufficient conditions in the problem of Lagrange with one variable end point.*

In the problem of Lagrange one considers an integral $\int f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$ and a set of differential equations $\phi_\alpha(x, y_1, \cdots, y_n, y_1', \cdots, y_n')=0$, $(\alpha=1, \cdots, m<n)$. The functions $y_1, \cdots, y_n$ are to satisfy the differential equations $\phi_\alpha=0$ and to render the integral an extreme value. In this paper the values of the functions $y_1(x), \cdots, y_n(x)$ are considered fixed for x_1, while the values for x_2 are restricted by a set of equations $x_2=X(t)$, $y_{12}=Y_1(t), \cdots, y_{n2}=Y_n(t)$. In this paper is given a set of sufficient conditions which when satisfied by the functions $y_1(x), \cdots, y_n(x)$ will cause them to minimize the above integral and satisfy the set of differential equations.

28. Professor L. H. McFarlan: *Problems of the calculus of variations in several dependent variables and their derivatives of various orders.*

The problem of minimizing an integral whose integrand contains x and n dependent variables $y_i (i=1, \cdots, n)$ and their derivatives $y_i', y_i^{(2)}, \cdots, y_i^{(k_i)}$, $(k_i \geqq 0)$, is treated by means of the known theory of the problem of Lagrange in the calculus of variations. Necessary and sufficient conditions on the minimizing arc analogous to those of Legendre, Jacobi, and Weierstrass in the more simple problems are derived. These are readily seen to contain as special cases the results obtained by Forsyth (*Calculus of Variations*, 1927) in which one dependent variable and derivatives of the first and second order enter the integrand. In the most simple problem of the calculus of variations with one dependent variable and its derivative of the first order, one passes from the fixed end-point case to problems in which the end points are required to lie on certain curves in the plane. In this paper the values of the functions y_i and their derivatives up to and including those of order k_i-1 for the values x_1 and x_2 defining the end points of the minimizing arc, are restricted by a system of differential equations of order k_i-1 in the variables y_i. The number of these differential equations is expressible in terms of n and the numbers k_i. For this case one obtains a necessary condition corresponding to the usual transversality condition. For special choice of the differential equations the focal point property is also generalized.

29. Professor W. E. Milne: *On the degree of convergence of the Gram-Charlier series.*

In a previous communication the author has given certain sufficient conditions for an assigned degree of convergence of the series

$$f(x) = \sum_{m=0}^{n} c_m e^{x^2/4}[(d/dx^m)e^{-x^2/2}] + R_n(x).$$

It is now shown that the following conditions are necessary: if $R_n(x) = O(n^{-k-\epsilon})$ for every n, where k is an integer and $\epsilon > 0$, then $f(x)$ must have a continuous derivative of order $2k$; if $R_n(x) = O(n^{-k})$ uniformly in the infinite interval, then $f(x) = O(x^{-2k})$ when x is large. It is also shown that if $f(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition and if $f(x)$ and its derivatives vanish to a suitable order at infinity, then the constants c_m can be chosen so that $R_n(x) = O(n^{-k/2})$.

30. Professor T. A. Pierce: *On an algorithm and its use in approximating roots of algebraic equations.*

If in the generating equations of a continued fraction the operation of addition be replaced by multiplication and that of division into unity be replaced by subtraction from unity we obtain a new algorithm. The expansion by this algorithm of any rational number is periodic. Non-periodic expansions represent irrational numbers. The application of the algorithm in approximating roots of algebraic equations is simple.

31. Mr. J. H. Roberts: *Upper semi-continuous collections of continua some of which are unbounded.*

In an abstract of a paper presented to the Society on December 27, 1928,

I failed to mention a certain hypothesis in the statement of certain results. The corrected statement is as follows: "If G is an upper semi-continuous collection of continua *no one of which separates the plane* S and such that (1) every point of the plane S belongs to some continuum of G, (2) at least two continua of G are unbounded, and (3) the space of elements of G is metric, then the space of elements of G is in continuous one-to-one correspondence with a subset of the plane S. In the present paper the condition that no continuum of G separates S is removed and the space of elements of G is shown to be a subset of a cactoid. A cactoid—as defined by R. L. Moore—is a bounded continuous curve M lying in space of three dimensions and such that (a) every non-degenerate maximal cyclic subset of M is a simple closed surface and (b) no point of M lies in a bounded complementary domain of any subcontinuum of M.

32. Mr. J. H. Roberts: *On a problem of Menger concerning regular curves.*

In 1927 Menger raised the question as to whether or not, for every positive number e, every regular curve M is the sum of a finite number of continua of diameter less than e, the common part of any two of which is at most one point. In the present paper an example is given which shows that the answer to Menger's question is in the negative. It is proved that for the special case where the set of points of M of order greater than 2 is null-dimensional Menger's question can be answered in the affirmative

33. Professors H. P. Robertson and Hermann Weyl: *On a problem in the theory of groups arising in the foundations of infinitesimal geometry.*

The paper appears in full in the present number of this Bulletin.

34. Dr. N. E. Rutt: *On the structure of certain plane continua.*

Let x denote a point and Z a bounded plane continuum which is the sum of a set $[X_\alpha]$ of continua each of which contains the point x, each of which remains connected when x is subtracted from it, and no two of which have any point in common except x. If none of the continua of $[X_\alpha]$ has a bounded complementary domain containing a point of any other continuum of the set $[X_\alpha]$, order is easily assigned among the elements of $[X_\alpha]$, so that with respect to some reference element precedence of elements of $[X_\alpha]$ and clockwise sequences may be defined. The following facts concerning the relations possible among the continua of $[X_\alpha]$ are established. If $[X_i]$ is a clockwise sequence of elements of $[X_\alpha]$, then no element X_a of $[X_\alpha]$ preceding all the elements of $[X_i]$ contains any limit point of $[X_i]$ except x. If $[X_i]$ is a clockwise sequence of elements of $[X_\alpha]$, and if both X_a and X_b are elements of $[X_\alpha]$ preceding all the elements of $[X_i]$, then of the two prime ends of X_a+X_b containing x which are arc-wise accessible from the unbounded complementary domain of X_a+X_b the one which is not a limit of any element of $[X_i]$ is also not a limit of the set of elements $[X_i]$.

35. Dr. Morgan Ward: *The foundations of general arithmetic.*

This paper is the first part of an amplification and extension of two short notes on the same topic (Proceedings of the National Academy, vol. 13 (1927), pp. 748–749; vol. 14 (1928), pp. 907–911). By taking as a basis a set closed under logical multiplication instead of a semi-group closed under ordinary multiplication, a postulational definition of an abstract arithmetic is obtained which insures unique factorization whether the set contains a finite or an infinite number of elements. Furthermore all the classical arithmetics (e.g., ideals of an algebraic field) are instances of this abstract system. A detailed development of the formal properties of the system discloses many analogies with the Boolean algebra of logic. In fact, every finite arithmetic may be interpreted as a set of elements in a Boolean algebra, although the converse theorem is false. Finally the case where the introduction of ideal elements is necessary is studied from the same abstract viewpoint.

36. Professor Hermann Weyl: *On the foundations of general infinitesimal geometry.*

The paper appears in full in the present number of this Bulletin.

37. Professor G. T. Whyburn: *Note on simple closed curves.*

In this paper it is shown that a simple closed curve (in n-space) may be defined as a compact continuum every point of which separates it locally into exactly two components; and in the plane it may be defined as a compact continuum every point of which is accessible from exactly two complementary domains. If these continua are not compact, they are open curves. It is also pointed out that if every point of a continuum M separates it locally into exactly k components, (k finite and >1), then $k=2$, and M is a simple closed curve or an open curve according as it is or is not compact.

38. Professor G. T. Whyburn: *On the set of all cut points of a continuous curve.*

In this paper it is shown that as a consequence principally of the author's results on the structure of a continuous curve relative to its cyclic elements it follows that the set of all cut points of any continuous curve in a metric space is homeomorphic with a certain kind of subset of a plane acyclic continuous curve. It is also shown, conversely, that a subset of this kind of any acyclic continuous curve in a euclidean space of dimension >1 is always identically the set of all cut points of some continuous curve. Thus we obtain a complete characterization of a point set which is topologically equivalent to the set of all cut points of a continuous curve. Incidentally, the results of this paper answer a question raised by C. Zarankiewicz in the Bulletin de l'Académie Polonaise, 1926, p. 362.

39. Professor W. M. Whyburn: *Functional properties of solutions of differential systems.*

In a recent paper (Annals of Mathematics, vol. 30 (1928), pp. 31–39) the author established existence and uniqueness theorems for a general class of differential systems. Using the notation of that paper, the present paper interprets the notion of the solutions of the differential systems being continuous functionals of the functions $f_i(x, y_1, \cdots, y_n)$, $A_{ij}(x, y_1, \cdots, y_n)$, $i, j = 1, \cdots, n$, and proves that under the hypotheses of the above-mentioned existence theorems the solutions of the systems will be continuous functionals of this type. The results here include the well known theorem concerning the continuity of the solutions of linear systems as functionals of the coefficients. The results of the present paper are useful in a study of the effects produced by small perturbations on dynamical systems.

40. Professor W. M. Whyburn: *On the critical sets for functions of n real variables.*

In a former paper, (*Non-isolated critical points of functions*, which appears in the present issue of this Bulletin (701–708), the author defined *critical sets* for functions of n independent real variables in cases where the critical points of these functions were not isolated. *Minimal sets* were characterized and theorems concerning the distribution of these were proved. The present paper continues this classification of critical sets to define several additional classes and extends the former theorems to cover this classification. The definitions and extensions are shown to be consistent with the classification by Morse (Transactions of this Society, vol. 27 (1925), pp. 345–396) when the critical points are isolated.

41. Professor W. M. Whyburn: *Functional inequalities.*

Inequalities of the types $F(x) < \sum_{i=1}^{n} f_i(x)g_i(x)$, $F(x) \leqq \sum_{i=1}^{n} f_i(x)g_i(x)$ where the functions involved are non-negative and continuous on the open or closed, finite or infinite interval X of the real variable x, are studied with a view to determining continuous functions $h_i(x)$ such that $0 \leqq h_i(x) \leqq g_i(x)$, $F(x) = \sum_{i=1}^{n} f_i(x)g_i(x)$ on X, and these functions take on assigned boundary values at points of given subsets of X. Cases where measurability and summability replace continuity in the above statements are treated. Functions of m real independent variables are considered. Applications to proofs of properties of differential equations are given.

42. Professor R. L. Wilder: *Characterizations of continuous curves that are perfectly continuous.*

It was shown by R. L. Moore and the author (see this Bulletin, vol. 29, p. 294) that in order that a continuum M should be a continuous curve, it is necessary and sufficient that every component of an open subset of M should be arcwise connected. The present paper considers those continuous curves which have the property that all of their subcontinua are continuous curves—such curves are called, for the sake of brevity, *perfectly continuous*, or *perfect continuous curves*. It is shown that the above property characterizing continuous curves may be modified so as to characterize perfect continuous curves. If M is any point set, and K is an F_σ which is a subset

of M, then let us call $M-K$ a *quasi-open subset* of M. Then in order that a continuum M in E_n should be a perfect continuous curve, it is necessary and sufficient that every component of a quasi-open subset of M should be arcwise connected. Incidentally, it is shown that every connected subset of a perfect continuous curve in a locally compact metric space is connected im kleinen—a property which also obviously characterizes these curves.

43. Dr. A. R. Williams: *On a type of transformation connected with a pencil of curves of order n.*

The fact that the n^2 base points of a pencil of curves of order n are determined by $n(n+3)/2-1$ of them gives rise to transformations of the plane. It is the purpose of this paper to discuss the locus of the remaining base points when a given number of them are fixed and the others necessary to determine the pencil are taken consecutive on some curve. Use is made of rational surfaces and developables in space of 3 and higher dimensions.

44. Dr. B. C. Wong: *A certain linear ∞^r-system of r-ic hypersurfaces in r-space.*

The hypersurfaces of order r passing through $r+1$ given $(r-2)$-spaces in S_r form a linear ∞^r-system $|V|$. The composite base manifold of $|V|$ is of order $r(r+1)/2$ and is composed of the $r+1$ given, $(r-2)$-spaces and the locus of the ∞^{r-2} lines incident with them. This locus is of order $(r+1)(r-2)/2$. It is shown that any r hypersurfaces of $|V|$ meet in only one distinct point and that, therefore, each hypersurface can be represented rationally on a hyperplane of S_r. It is possible to find r hyperquadric surfaces that will effect the transformation by making any point and the point of intersection of its polar hyperplanes with respect to the r hyperquadric surfaces correspond to each other.

45. Dr. B. C. Wong: *An octavic hypersurface in space of four dimensions.*

This paper deals with the hypersurface V_{r-1} of order 2^{r-1} in r-space represented by the equation $x_0^{1/2}+x_1^{1/2}+\cdots+x_r^{1/2}=0$ and, in particular, with the octavic hypersurface V_3^8 in 4-space. V_{r-1} is the generalization to r-space of Steiner's quartic surface in 3-space. It is touched by each of the $\binom{r-1}{k+1}$ k-spaces $[0<k<r]$ of the coordinate simplex along 2^{r-k-1} coincident $(k-1)$-dimensional varieties of order 2^{k-1}. The representation of V_3^8 and that of its reciprocal hypersurface on a hyperplane are discussed and also that of its general hyperplane section.

46. Professor D. W. Woodard: *On a system of axioms for two-dimensional analysis situs.*

In the author's doctoral dissertation, *On two-dimensional analysis situs with special reference to the Jordan curve theorem*, published in volume 13 of Fundamenta Mathematicae, the following axiom is used: *If P is a point on the boundary of a region R, there exist two simple continuous arcs AP and*

BP such that $AP-P$ and $BP-P$ are subsets of R and $S-R'$ respectively. Here S represents the set of all points in the space under consideration and R', the set of points contained in R and its derivative. It is proved that the axiom stated above may be replaced by the following weaker axiom: *Every point on the boundary of a region is a limit point of the exterior of the region.* In connection with this axiom an independence example is provided. All of the results of the paper follow from the modified system of axioms. In fact, it is necessary only to make certain additional arguments in the proof of Theorem 7 of the paper.

47. Dr. Leo Zippin: *On acyclic continuous curves.*

The author constructs a plane, bounded, acyclic continuous curve which has a subset homeomorphic with any given acyclic continuous curve. In case the given curve is unbounded it is homeomorphic with the complement in a bounded acyclic continuous curve of a subset of end points. He proves, also, that if B is a closed and totally disconnected subset of a continuous curve C, in order that B be the set of end points of an acyclic continuous subcurve of C it is necessary and sufficient that if M is any bounded continuous curve of C and D a closed and totally disconnected subset of interior points of M, then $M-D$ is connected.

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THE PROBLEM OF THE CUBIC VARIETY IN S_4*

BY VIRGIL SNYDER

1. *Introduction.* One of the outstanding problems of algebraic geometry that is still unsolved is the classification of Cremona transformations in space of more than two dimensions, and another, closely related to the preceding, is that of possible series of composition.

The theorem of Noether (80), later clarified and interpreted by Segre (90), Castelnuovo (14), Chisini (17), and Alexander (1), has answered the latter problem for the plane; while a large number of papers, represented in particular by the recent work of Hudson (67), Mlodzieioski (73)–(76), and Montesano (79), have made the construction of tables a definite procedure.

The smaller problem of the involutorial transformations has been completely solved by Noether (81), Bertini (6), and others, thus furnishing a weapon of incalculable importance for the study of various applications. Periodic transformations and their groups have been studied from various points of view by Kantor (69)–(70), Wiman (112), and Coble (20)–(21).

The corresponding problem in space is still almost undeveloped. Notwithstanding the excellent report made by Coble (22), the less ambitious one by me (103), the appearance of the extensive treatise by Hudson (67), and the Report of a committee of the National Research Council (106), I wish to speak of one phase of the latter problem. For regular transformations in space of three dimensions, the important theorem of Hudson (66) and a recent memoir of Montesano (78) are distinct steps in advance. When the

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defining elements are chosen independently, without contact conditions, the table of characteristics for a Cremona transformation and for its inverse has been completely constructed.

I wish to confine my attention, however, to the part taken by the cubic variety of four-way space S_4 in the study of involutorial birational transformations of S_3.

The study of the involution defined by the web of quadrics through six points leads at once to the Weddle surface, the locus of the vertex of the quadric cones through the six points, as surface of coincident points. If the quadrics of the web be made projective with the planes of another space S_3', a (1, 2) correspondence is established, and the Kummer surface appears as surface of branch points in S_3'. All the properties of the general transformation and of all particularizations can be made from this point of view. This study of (1, 2) correspondences is an extension of that initiated by De Paolis (27) for S_3, and applied to double planes by Enriques (32) and by Castelnuovo and Enriques (15), thus confirming the result obtained by Noether (81) concerning the number of plane involutions of order two.

But the Kummer surface is the apparent contour of the cubic variety with ten nodes, as has been shown by Segre and further developed by Hudson (68) and by Snyder (101). Moreover, the same surface also appears in a similar rôle in connection with involutions defined by webs of cubic and of certain quartic surfaces. See Sharpe and Snyder (100).

In the same way, the quartic surface with less than sixteen nodes, birationally equivalent to the focal surface of line congruences of order two and class 3, 4, 5, 6 appear as surfaces of branch points of certain involutions, and each is the apparent contour of a cubic variety with one or more actual double points. Thus, to a point on the cubic variety correspond a pair of points in S_3, each point of which uniquely determines the other. These pairs of points determine an involution belonging to V_3.

2. *Rational Involutions.* Given an algebraic curve C in S_n, a necessary and sufficient condition that C is rational is that its points can be put into (1, 1) continuous correspondence with those of a straight line. (Definition.)

If $\rho x_i = f_i(y_1, y_2)$, $(i = 0, 1, \cdots, n)$, define the coordinates (x) of a point on $C(x)$ and y_1, y_2 those of the image point on the line, then f_i is a rational integral polynomial homogeneous in y_1, y_2.

If several sets of values of y_1, y_2 define the same variable point (x), then functions $y_1 = \phi_1(y_1', y_2')$, $y_2 = \phi_2(y_1', y_2')$ can be found such that the relation between (y') and (x) is (1, 1). This is expressed by saying that *every involution on a rational curve is rational.* (Theorem of Lüroth (71)).

Similarly, given an algebraic surface $F(x)$ in S_n, such that $\rho x_i = f_i(y_1, y_2, y_3)$, wherein each f_i is a rational integral form of the same degree in (x), then $F(x)$ is rational. If several sets of values of (y) define the same variable (x), then functions $y_i = \phi_i(y_1', y_2', y_3')$ can be found such that the relation between (y') and (x) is (1, 1). Hence *every involution on a rational surface or in particular on a plane, is rational.* (Theorem of Castelnuovo (11).) The theorem for involutions of order two was already known. See Noether (81) and Bertini (6). More recent contributions to the classification of plane involutions of order higher than two have been made by Howe (65), Hollcroft (64), and Sharpe (99).

Given a three-dimensional manifold $M_3(x)$ and $\rho x_i = f_i(y_1, y_2, y_3, y_4)$ a parametric representation, wherein f_i is a rational integral form, we cannot conclude whether M_3 is rational or not. The condition is necessary but it is not sufficient. If various sets of values (y) define the same variable set (x), it may not be possible to find a set of functions $y_i = \phi_i(y_1', y_2', y_3', y_4')$ such that the relation between (x) on $M_3(x)$ and (y') in S_3' is (1, 1). (Theorem of Enriques (36).)

Illustrations have been given of irrational involutions in S_3 belonging to a three-dimensional variety; others were furnished by Aprile (2) but all are of order larger than two. No illustrations have been found of three-dimensional in-

volutions of order two that are known to be irrational.

In particular, it is not known whether the points of the general cubic variety V_3 in S_4 can be put into (1, 1) correspondence with those of S_3.

It is known that an involution of order two in S_3 belongs to it. The existence of this involution was known to Noether, and later proved by Segre (89) and by Marletta (72). The actual form of the parametric represention is

$$\begin{aligned}
\rho x_1 &= y_1^2(2y_1^3 y_3 y_4 + Ay_3 + By_1),\\
\rho x_2 &= y_1 y_2(\quad\quad\ \ " \quad\quad),\\
\rho x_3 &= -\, y_3^2(\quad\quad\ \ " \quad\quad),\\
\rho x_4 &= y_1(y_1^3 y_3 y_4^2 + By_1 y_4 - Cy_3),\\
\rho x_5 &= -\, y_1(y_1\, y_4^2 + Ay_1 y_4 + Cy_1),\\
V_3 &= x_1 x_4^2 + x_3 x_5^2 + ax_4 + bx_5 + C = 0,
\end{aligned}$$

$$a = a_0 x_3^2 + a_1 x_3 + a_2, \qquad A = a_0 y_3^4 - a_1 y_3^2 y_1 + a_2 y_1^2,$$

$$b = b_0 x_3^2 + b_1 x_3 + b_2, \qquad B = b_0 y_3^4 - b_1 y_3^2 y_1 + b_2 y_1^2,$$

$$c = c_1 x_3^2 + c_2 x_3 + c_3, \qquad C = c_1 y_3^4 - c_2 y_3^2 y_1 + c_3 y_1^2,$$

where a_i, b_i, c_i are binary forms of order i in x_1 and x_2.

This form of order 7 is the simplest one consistent with a quadratic locus of invariant points. This result was obtained and the associated involution of order 6 is described by Snyder (104).

3. *Involutions Belonging Multiply to a Line Complex.* Every V_3 contains ∞^2 straight lines. See Enriques (28) and Fano (57)–(58). Let l be a line on V_3, and P any point on V_3.

The plane (P, l) meets V_3 in a residual conic through P which meets l in two points Z_1, Z_2. The line PZ_i is tangent to V_3 at Z_i. Conversely, any line tangent to V_3 at a point Z on l meets V_3 again in a point P.

Between the points P of V_3 and the tangents at points on l there is therefore a (1, 2) correspondence. These tangent lines form a special linear line complex which can be mapped birationally on the points of an S_3. The lines joining a pair of associated points PP' in S_3 always meet the rectilinear image

of l; hence they define a special linear complex. But every line of the complex contains two pairs of associated points. (Snyder (104).)

Other I_2 in S_3 exist which are contained multiply in a rational complex. The simplest one is obtained by constructing a $(1, k)$ correspondence between a pair of lines on a regulus and the quadrics of a pencil. A point P of S_3 determines a quadric passing through it, and hence a pair of directrices (d, d') of the regulus. The transversal of (d, d') through P meets the quadric in P'. The line PP' contains k pairs of associated points, and belongs to a general linear complex. (Snyder (107).)

Consider a line l and a pencil of surfaces of order n having the line to multiplicity $n-2$. Make the same $(1, k)$ correspondence between the points of the line and the surfaces of the pencil. The line may be replaced by a conic provided the surface is of order less than 5, or by a space cubic.

A pencil of quadrics and a rational curve of any order may be taken. Similarly, a linear system consisting of a line and of a rational curve of order m meeting it in $m-1$ points may be put in $(1, k)$ correspondence with the quadrics of a pencil. Either the line or the curve may remain fixed. The system may be replaced by a pencil of cubic curves as directrices. All of these forms contain types which belong to the cubic variety as particular cases. Incidentally they all possess properties not shared by other involutions heretofore known.

Another illustration is that of order 7 found by Montesano (77). Given five linear line complexes K_i and a projective form q or g of five elements. An arbitrary plane π of space has five poles O_i as to K_i, and in it is one point O' such that the lines $O'O_i$ $\overline{\wedge} g$. There is one and only one complex K in involution with every K_i. The plane π has a pole O as to K. The relation between O, O' is birational and involutorial.

The line OO' describes a linear complex K and every line of K contains three pairs of conjugate points. There is no surface of invariant points, but a curve of order 8 and genus 5, birationally equivalent to the only fundamental curve of

the system, apart from parasitic lines. It is not known whether this involution belongs to the cubic variety or not. Its properties are different from those of the other types just mentioned.

In particular, if the set of complexes be considered, such that each new one is in involution with five of the given ones (K_i, K), and the corresponding transformation be considered, the rôles of the fundamental curve and of the curve of invariant points will be interchanged. The scheme really furnishes ∞^2 pairs of such complementary involutions, corresponding to the ∞^2 possible values of g.

4. *Infinite Discontinuous Groups.* Various infinite discontinuous groups which belong to V_3 exist. We shall consider six distinct generating operations numbered (a), (b), (c), (d), (e), (f) below. It has not been shown whether others also exist.

(a). *The projection of V_3 upon itself from a vertex A upon it.* The product AB of two such projections is not periodic, nor can it be replaced by another product. Similarly for the product of three or more such projections.

The projection A is Cremonian, and quadratic. An S_3 is transformed into a quadric variety having the polar quadric of A for locus of invariant points. The hyperplanar sections of V_3 are transformed into sextic surfaces all passing through the six lines of V_3 which pass through A. These lines have the property that the image of a point on any one of them is the whole line passing through it.

In the product AB the S_3 sections of V_3 are transformed into surfaces of order 12, having six double basis lines through B, and six conics, all lying on a quartic surface, through another point, the image of A in the operation B. Similarly for products of more projections $A, B, C \cdots$. (Snyder (102).)

(b). *Let l be a fixed line on V_3. A point P of V_3 and l determine a plane which meets V_3 in a residual conic passing through P.*

The line l has a pole L as to this conic. The harmonic homology defined by L and l leaves the conic, and hence also V_3, invariant. The transformation is Cremonian, of order 8. The conjugate of an S_3 contains l to multiplicity 7. The locus of invariant points is a variety of order 5, containing l to multiplicity 4. The image of l itself is a variety of order 7, containing l to multiplicity 6. Every S_3 through l is transformed into itself.

There are ∞^2 lines on V_3, each of which has an associated involution (b). The product of two such involutions, associated with different lines, is not periodic, nor can one such product be replaced by another when the lines are taken arbitrarily.

The product of a transformation of the type (a) and a transformation of the type (b) is in general not periodic. (Snyder (102).)

(c). *Consider a pencil of* S_3 *having* π: $(ax)=0$, $(bx)=0$ *for base, and the line* $l \equiv x_1=x_2=x_3=0$ *on* V_3. Make the points of the line and the hyperplanes of the pencil projective, by associating each S_3 with the point P in which it meets l. Project the cubic surface S_3, V into itself from P. The transformation is Cremonian, of order 4, the fundamental elements are l taken simply, and the basis plane taken twice. The locus of invariant points is a cubic variety containing l, π each simply.

A transformation of this kind exists for every line l on V, and when l has been fixed, the basis plane may be chosen in ∞^6 ways. The product of two or more transformations of this type is not periodic. The product of any transformation of type (c) with one of type (a) or of type (b) or any sequence of (a), (b), (c) is in general not periodic.

The line l may be replaced by a conic, provided the basis plane has one point on it. The variety V_3 contains ∞^4 conics. Moreover, a space cubic with two points on the basis plane or a rational quartic with all three basis points upon it, may be taken. The directrix curve appears as a basis curve with multiplicity equal to its order in every case.

(d). *Assume an arbitrary fixed line m through a point M on V_3, but not lying on V_3.* A point P of V_3 and the line m determine a plane which meets V in a cubic curve through M. The first tangential M' of M, joined to P, meets V_3 in P'. The transformation P, P' is involutorial and Cremonian of order 8. The locus of M' is the surface of intersection of V_3 and the three-dimensional tangent space at M, of order 3. Every plane through m is transformed into itself. The transformation in each such plane is quadratic, a perspective Jonquières type of the third kind. The fundamental line is the tangent to the cubic curve at M', and the locus of invariant points is the polar conic of M' as to the cubic. The images of the hyperplane sections $|x|$ of V_3 are surfaces of order 24, $|8x|$, having M to multiplicity 6, and the six lines of V through M to multiplicity 2. They also contain the locus of the first tangential of M' as the plane describes the linear system through m.

The product of two transformations of type (d) is not periodic. There are ∞^3 points on V_3, and ∞^3 directions m associated with each point. The properties of this transformation should be further studied, both in S_4 and in S_3. Many of them are different from those of involutions heretofore known. The product of a transformation of type (d) with any one of a preceding type or one derived from any combination of the preceding types is in general not periodic.

(e). *Let l, l' be two skew lines on V_3 and π a plane not meeting either.* The plane is the basis of a pencil of S_3. The S_3 through a point P meets (l, l') in (K, K'). The line (K, K') meets V in M. The line PM meets V_3 in P'. The transformation is involutorial, and consists in interchanging P and P'. The locus of M is a rational C_4. The lines l and l' lie in a space Σ_3 which cuts V_3 in a cubic surface F_3 containing l and l'. The pencil (S_3) meets (l, l') in ranges projective with it, hence with each other, hence (K, K') describes a quadric surface H_2 in Σ_3, through (l, l'). The residual intersection of F_3 and H_2 is a rational C_4 which meets (l, l') each in three points.

Either or both lines may be replaced by conics skew to each other, provided π meets each conic in one point, or one may be replaced by a cubic with two points on π, with the other directrix either a line skew to π or a conic with one point on π. When both directrices are cubics, they must have a common point on π. Corresponding forms exist in a three-way space which leave a given cubic surface invariant, each of which presents interesting configurations of fundamental elements.

(f). Another transformation is the following. *Given a line l on V_3 and a plane π of S_4, skew to l.*

The points O of π and the planes through l are projective, and perspective. The plane through l meets V_3 in a residual conic. Consider the harmonic homology having O for center and its pole as to the conic in its associated plane for axis. This procedure defines an involutorial birational transformation which leaves each conic of l in a plane through l invariant, and hence transforms V_3 into itself.

It is Cremonian. The line l and the plane cubic curve section of V_3 by π are the fundamental elements. The plane π may be replaced by any rational surface meeting an arbitrary plane through l in one point apart from those on l.

5. *Continuous Groups.* A curve of genus zero belongs to a three-parameter group of continuous transformations. One of genus 1 has ∞^1 transformations which do not form a group. If the genus exceeds unity, the curve belongs at most to a finite group. Surfaces invariant under ∞^1 linear transformations are rational or reducible to ruled surfaces. If they are invariant under ∞^2 such transformations they are necessarily rational. (Fano (40).) Every algebraic surface, invariant under a transitive continuous group of linear transformations, can be transformed birationally into a plane, or a quadric of S_3, or into a rational cone normal in a certain S_{m+1} in such a manner that the group considered gives rise respectively to a group of plane homographies, or to a group of homographies in S_3, or in S_{m+1}, each leaving the associated form invariant. (Fano (41) and (44).)

Every three-parameter linear group of S_3 leaves a series of forms generated by planes invariant, none containing a simpler one. (Fano (40).)

The three-dimensional varieties of S_4 invariant under ∞^4 continuous linear transformations are classified into five types, which do not include the general cubic variety. (Fano (43).)

Surfaces invariant under a continuous group of Cremona transformations are all reducible to rational surfaces in S_3 invariant under linear transformations. (Fano (45).) Similarly for three-dimensional varieties. (Fano (46).)

Given a homogeneous linear differential equation of order n. If a system of solutions $y_1, \cdots, y_n$ be regarded as projective coordinates of a point of S_{n-1}, the point will describe a rational curve when the coefficients in the given equation are rational functions of x. If the (y) satisfy one or more algebraic relations, the curve will lie on a corresponding number of varieties. The group of rationality of the equation leaves each variety of the system invariant. Finite groups of Cremona transformations in the plane, both according to the order and to the type, in the sense of Kantor and of Wiman have all been determined in this way. (Enriques and Fano (38).) The types of continuous groups in the plane are projective, quadratic and Jonquières, and in S_3 contain these and two other ∞^3 types. The generalized Jonquières groups leave a pencil of planes or a bundle of lines (or both) invariant. (47).

The imprimitive groups of continuous Cremona transformations in S_3 belong as subcases in the types derived by Enriques (29) and Fano (46), (48).

The complete enumeration of continuous Cremona transformations in S_3 can be made by starting from the generating infinitesimal transformation. This has been done and the preceding results confirmed. (49).

A differential equation of the type in question is always solvable when more than one algebraic equation is identically satisfied by the solutions. (50).

The group of rationality may leave a quadratic form in S_{n-1} invariant. (52). For $n=6$, this case can be interpreted in terms of lines of S_3; instead of a curve, the locus is now a ruled surface. For $n=5$, the same interpretation can be made by assuming a linear relation among six solutions. The ruled surface is then contained in a linear line complex. (51). If the discriminant of the quadratic form vanishes (special complex), the equation can be reduced to one of order 4. (54).

Every M_3 invariant under ∞^3 projective transformations is rational. If the group is not integrable, the existence of ∞^1 rational surfaces on it insures its rationality. (53).

These results have been collected and systematically developed by Fano (55). It is shown that the existence of algebraic relations among the fundamental solutions of a linear differential equation is a necessary and sufficient condition for the vanishing of certain invariants belonging to the equation.

On the general quartic three-dimensional variety V_3^4 of S_4, without double points, the system of hyperplane sections is the only linear system of regular surfaces with genera equal to unity and of dimension not less than 2. This variety is not invariant under any birational transformation, with the possible exception of linear ones.

The M_3^6 of S_5, complete intersection of V_4^2 and V_4^3, does not contain such surfaces either. Every birational transformation which leaves it invariant is the product of a finite number of double projections from lines lying on it and of possible linear transformations. (Fano (61).)

6. *Invariants.* The question naturally arises whether algebraic or topological invariants exist, particular values of which can indicate the rationality or irrationality of V_3.

A necessary and sufficient condition that an irreducible algebraic curve in space of any number of dimensions shall be rational is that its genus is zero. (Clebsch (18).)

An algebraic surface has an infinite number of invariants analogous to the genus of a curve, but a necessary and

sufficient condition that a surface shall be rational is that its arithmetic genus p_a and its bigenus P_2 should both be zero. (Castelnuovo (13).) That a surface may have both its geometric genus and its arithmetic genus equal to zero and still not be rational, was shown by Enriques (35).

A system of invariants of three-dimensional varieties was determined by Pannelli (84), analogous to the genera of a surface. (Castelnuovo and Enriques (16).) Some of the fundamental concepts of invariants on hyperdimensional manifolds had been discussed by Noether (82). It was then shown by Severi (97) that all these invariants vanish for the cubic variety of S_4, also for the general quartic V_3^4 of S_4, and for the $M_3^6 \equiv (V_4^2, V_4^2)$ of S_5, hence that the various indices of irregularity are zero. These latter V_3^4, M_3^6 contain only complete intersections. (Severi (92), Fano (59).)

The algebraic surfaces contained in V_3^3 were obtained by Fano (56).

The M_3^6 is the representative of the general cubic line complex of S_3, which has been extensively studied by Voss (111) and by Veneroni (109). In particular, it has ∞^1 linear pencils of lines or quadric reguli, hence M_3^6 contains ∞^1 straight lines. By projecting the manifold from one of them, it can be mapped on S_3 doubly, with a sextic surface of branch points. The two forms, V_3^4 and M_3^6, are birationally distinct. (Fano (59).)

The assumption that either contains a homaloidal system of surfaces leads to a contradiction; hence both of these forms are irrational, that is, they can not be mapped birationally on S_3. (Fano (59).) However, both can be mapped on S_3 by means of an involution that is rational in one sense only.

The actual process was outlined for M_3^6 by Enriques (36), which establishes the important result that irrational involutions in S_3 exist. Other examples of involutions belonging to three-dimensional varieties were given by Marletta (72), and Aprile (2) showed that the Enriques involution is of

order 216, but that the manifold can be mapped on S_3 by means of an involution of order 36. Further conditions for rationality were found by Enriques (37).

These forms are much more general than V_3^3. The general M_3^6 does not contain any planes, since the general cubic line complex of S_3 does not contain a bundle of lines. If now the complex be particularized to contain three distinct bundles of lines (or dually, three plane fields of lines), then the M_3^6 contains three planes belonging to the same system on the quadric variety passing through it. This particularized M_3^6 is birationally identical with the general V_3^3.

Similarly, the V_3^4 can be particularized to the extent of having a double line and still be much more general than the general cubic variety. It may be desirable to study the properties of these particular forms in order to obtain those of the general V_3^3.

The two general forms V_3^4 and M_3^6 are included in the general category M_3^{2p-3} of S_{p+1} (Fano (62)) having surface sections regular with all their genera equal to unity, and curve sections canonical curves of genus p. V_3^4 corresponds to $p=3$ and M_3^6 to $p=4$. The case $p=2$ appears as a double S_3 with a sextic surface of branch points, but this can be obtained by projecting M_3^6 upon S_3 from one of its lines, in a particular form.

The case $p=5$ is represented by the intersection of three general quadratic varieties in S_6. It contains no surfaces other than complete intersections with other forms. This manifold can be mapped on S_3 by an involution of order 4. It contains an infinite number of rational congruences of the first order, of rational curves.

The general $V_3^n : S_1^{n-2}$ of S_4 has been studied from various points of view, but most of the properties found for larger values of n do not exist for $n=3$. (Enriques (33).)

As yet no topological invariants have been found that are characteristic of the general V_3^3, normal in S_4.

Various properties regarding the analytical representation of a variety and of manifolds have been found by Severi

(91), those regarding postulation (93), those concerning complete intersections of non-singular forms (94), (95).

7. *Sections by* $|S_3|$. If all the plane sections of a surface in S_3 are rational, the surface is rational. It is either ruled or is a Steiner quartic surface with three concurrent double lines. (Picard (85), Guccia (63).)

If the sections by $|S_{n-1}|$ of a manifold M_3 in S_n are rational surfaces, M_3 is rational except possibly when M_3 is V_3^3 of S_4, concerning which no conclusion can be drawn, as the method does not apply. (Fano (60).)

Probably the transcendental methods of Bagnera-De Franchis (3)-(5), those of Enriques-Severi (3), and those of Severi (96), (98) as applied to hyperelliptic surfaces, may be extended to three-dimensional varieties. As yet no light has been thrown on V_3^3 from this source. Another possible method of extension is that of Comessatti (23) in his study of real rational surfaces.

8. *Rational* V_3^3. Let there be given a cubic variety $V_3^3(x) = 0$ in S_4, such that each x_i may be expressed in the form $\rho x_i = f_i(y_1, y_2, y_3, y_4)$, each f_i being a rational quaternary form of order N. When these values of x_i are substituted in $V(x) = 0$, it shall be identically satisfied, and that for an arbitrary set of values of the y_i; no other set can be found which will define the same value of (x).

Since $\sum a_i x_i = 0$, $V_3^3(x) = 0$ define a cubic surface, it follows that the $f_i(y)$ must satisfy the following properties.

(a) The system is linear of dimensionality 4.

(b) Any two surfaces of the system intersect in a curve of genus 1. (Plane sections of a cubic surface.)

(c) Any three intersect in three variable points.

From any simple point O_x on $V_3^3(x)$ project the variety on any space $S_3 \equiv \pi$ not passing through O_x. In π, every point is the image of two points of V_3^3; those on the generators of the cone of apparent contour project into the points of the quartic surface of branch points $L_4(x) \equiv u_2^2(x) - u_1(x) \cdot u_3(x) = 0$ in which $u_1 = 0$ is the tangent S_3 to V_3^3 at O_x. The plane

$u_1=0$ in π touches $L_4(x)=0$ along the conic $u_2=0$, which contains the six double points on $L_4(x)=0$, defined by $u_1=u_2=u_3=0$. Between the two three-spaces π and (y) there is a (1, 2) correspondence, in which $L_4(x)=0$ in π is the surface of branch points.

Let the equation of V_3^3 be $u_1x_5^2+2u_2x_5+u_3=0$. Then the projection of V_3^3 upon $x_5=0$ as double S_3 from $O_x\equiv(0, 0, 0, 0, 1)$ has the surface $L_4(x)=0$ of branch points. Let O_y be the image of O_x supposed to be non-singular in $\rho x_i=f_i(y)$. Then (y_0) satisfy the four equations $f_i(y)=0$, $(i=1, 2, 3, 4)$.

In the (1, 2) correspondence between (y) and $x_5=0$, the point O_5 is singular. In the projection upon $x_5=0$ from O_x, the conjugate of O_x is the plane $u_1=0$; it is the intersection with $x_1=0$ of the tangent S_3 to V_3^3 at O_x. The conjugate of O_y is then the f of the ∞^4 system defined by $u_1(f)=0$.

The points of $u_1=0$ in π are of three kinds, those not on L_4 are all conjugate to O_x; those on $u_2=0$, $u_1=0$ but not on $u_3=0$ are all branch points at O_x, as $u_2=0$ defines the quadric cone of inflexional tangents to V_3 at O_x.

Finally, the six double points $u_1=0$, $u_2=0$, $u_3=0$ are images in $x_5=0$ of the six lines of V_3 which pass through O_x.

Any S_3 passing through O_x meets V_3 in a cubic surface which is transformed into itself by interchanging the two points of V_3 on every line through O_x.

Hence in (y), each $f_i(i\neq 5)$ is transformed into itself. The curves $f_k=0$, $f_i=0$, $(i, k\neq 5)$ of genus 1 are the images of the lines of S_3.

The ∞^4 system $|f|$ is not invariant under I_2, and the subsystem is still of grade 3 when the fixed point O_y is adjoined.

A section $\sum a_ix_i=0$ not passing through O_x meets V in a general cubic surface. The three-dimensional cone joining the surface to O_x meets V in a sextic surface, the complete intersection of V and a quadratic variety $|2x|$, obtained from $\sum a_ix_i$ by the quadratic transformation

$$(T)\qquad \begin{cases} \rho x_i' = u_1x_i, & (i=1, \cdots, 4),\\ \rho x_5' = -u_1x_5-u_2. \end{cases}$$

The corresponding transformation in (y) becomes

$$\sigma f_i(y_1', \cdots, y_4') = u_1(f(y))f_i(y),$$
$$\sigma f_5(y_1', \cdots, y_4') = -u_1(f(y))f_5(y) - u_2(f(y)),$$

and for the involution I_2, of associated points,

$$f_i(y') = f_i(y).$$

Thus, the quadratic image of any S_3 section in the transformation T passes through the quadric surface $u_1=0$, $u_2=0$, which is the intersection of the tangent space to V_3 at O_x and the quadric polar of O_x as to V_3.

This is the quadric of inflexional tangents to V_3 at O_x. It meets V_3 in the six lines passing through O_x.

The images of these six lines are six curves through O_y, common to the conjugates of the ∞^4 $|f|$. They are all parasitic curves.

Suppose V_3^3 has a double point not lying in the tangent S_3 at the center of projection. Then in $x_5=0$ the surface of branch points $L_4=0$ has a double point not lying in the singular plane $u_1=0$ associated with the center of projection.

Conversely, if $L_4=0$ has a double point not lying in $u_1=0$, from the vanishing of the first derivatives of L_4 as to x_1, those of V_3^3 can be shown to vanish. Hence we may draw the following conclusion.

A necessary and sufficient condition that the surface of branch points $L_4=0$ has a double point not in the singular plane associated with the center of projection is that V_3^3 has a double point not in the tangent space at the center of projection.

If the values x_i are substituted in L_4, the result is a perfect square $K^2(y)=0$: the surface of contact of V and its three-dimensional tangent cone from O_x.

If V_3^3 has a double point P not at O_x, its image will be an actual double point of $L_4=0$, not on $u_1=0$. In this case, the parametric representation of $V_3^3:P^2$ can be obtained immediately by projecting V_3^3 from P and cutting the projecting cone by any S_3 not passing through P.

If $u_1 \equiv x_1$, and if u_2 and u_3 contain x_1 at most to the first power, then

$$V_3 \equiv u_1 x_5^2 + 2u_2 x_5 + u_3$$
$$= x_1 x_5^2 + 2(x_1 v_1 + w_2)x_5 + w_3 = 0 \equiv x_1 t_2 + t_3,$$

where v_i, w_i are ternary in x_2, x_3, x_4 of degree i, and

$$t_2 = x_5^2 + 2v_1 x_5 + v_2, \qquad t_3 = 2w_2 x_5 + w_3,$$

represents a cubic variety having a double point at the point (1, 0, 0, 0, 0).

The representation in the space (y) now takes the form

$$\rho x_1 = -t_3(y_2, \cdots, y_5),$$
$$\rho x_i = y_i t_2(y_2, \cdots, y_5), \quad (i = 2, \cdots, 5).$$

The image of the double point is the quadric surface $t_2 = 0$, and the fundamental element in (y) is the sextic space curve γ_6 of genus 4, $t_2 = 0$, $t_3 = 0$.

The ∞^4 linear system, images of the S_3 sections of V_3, now has the form

$$\sum a_i y_i t_2 - a_1 t_3 = 0;$$

thus it includes all the cubic surfaces passing through γ_6. This system satisfies conditions (a), (b) and (c).

The image in (y) of the vertex $(0, 0, 0, 0, 1) \equiv O_x$ is the point (0, 0, 0, 1), hence the ∞^3 subsystem which defines the (1, 2) correspondence between $x_5 = 0$ and (y) has the form

$$\rho x_1 = -2w_2(y_2, y_3, y_4)y_5 + w_3(y_2, y_3, y_4),$$
$$\rho x_2 = y_2 t_2(y_2, \cdots, y_5),$$
$$\rho x_3 = y_3 t_2(y_2, \cdots, y_5),$$
$$\rho x_4 = y_4 t_2(y_2, \cdots, y_5),$$

$$\sigma y_2 = x_1 x_2,$$
$$\sigma y_3 = x_1 x_3,$$
$$\sigma y_4 = x_1 x_4,$$
$$\sigma y_5 = -(v_1 x_1 + w_2) \pm [(v_1 x_2 + w_2)^2 - x_1(x_1 v_2 + w_3)]^{1/2}.$$

The equations of I_2 in (y) can now be determined at once. The transformation is quartic, monoidal perspective. The surface of invariant points is

$$w_2y_6^2 - w_3y_5 + v_1w_3 + v_2w_2 = 0.$$

Now suppose that V_3^3 is always rational, and that its equation contains a parameter λ, such that when $\lambda=0$ the corresponding variety acquires a double point at P. Then

$$\rho x_i = \phi_i(y,\lambda) = y_iv_2(y)F(y) + \lambda f_i(y,\lambda), \quad (i = 2, \cdots, 5),$$

$$\rho x_1 = \phi_1(y,\lambda) = -t_3(y)F(y) + \lambda f(y,\lambda).$$

Consider any linear system of at least ∞^4 rational surfaces of order N. By using the common basis elements as part or all of the fundamental elements of the transforming system, suppose the system has been reduced to its simplest form. Then within the system construct a subsystem which satisfies conditions (a), (b), (c). Evidently, by any birational transformation of (y), the properties of $V(x)$ will remain unchanged. Any rational surface may be transformed by means of its adjoints into one having plane sections of genus 0, 1, 2, 3 or of any positive genus, but hyperelliptic. (Enriques (33), (34).) We shall consider only those linear systems which belong to one or another of these five types, I, $\cdots$, V and which satisfy the conditions (a), (b), (c).

Finally, we select an additional basis point to define the (1, 2) correspondence. The five types will be considered in turn, but the three following properties will first be established.

9. *Properties of Basis Elements.* In the ∞^4 system $|f(y)|$, images of the spatial sections $|x|$ of V_3^3, is a subsystem ∞^3 $|\bar{f}(y)|$ having an additional basis point not a basis point of the larger ∞^4 system. This special basis point is the image of an ordinary point O_x on V_3^3. If now V_3^3 be projected upon a space π not passing through O_x from O_x doubly, then between π and (y) exists a (1, 2) correspondence, such that each $\bar{f}(y)$ is transformed into

itself, both images of a point in the image plane in π being in this surface. In the projection of V_3^3 from O_x upon π, the fundamental element is the plane, which is the intersection of the tangent S_3 to V_3^3 at O_x with π. The image of this plane in (y) is a definite $\bar{f}$ of the subsystem.

If $|\bar{f}|$, and therefore $|f|$, has any other simple basis point, there is one surface $\bar{f}$ belonging to it having this point for double point; the point and the nodal surface are conjugate in the associated I_2 of (y). The associated plane in π must therefore touch L at every common point, since its image in (y) is composite; that is, the plane is a singular tangent plane of L and hence contains six double points. In no cases can all these points be in the first singular tangent plane. Therefore we have the following results.

(A) *If $|f|$ contains a simple isolated basis point, not that defining $|\bar{f}|$, L has double points not in the characteristic singular plane.*

Similarly, if any surface of the system $|\bar{f}|$ is composite, the two components are conjugate to each other in I_2, and the associated plane in π must touch L at every common point; that is, it is a singular plane; hence L contains double points not in the singular plane u_1. (Sharpe and Snyder (100).)

(B) *If $|\bar{f}|$ contains a composite surface, L has double points not in the characteristic singular plane.*

Let δ be a curve in (y), which meets the surfaces of $|f|$ only in basis points or in points of basis curves. The surfaces of the system which pass through a point of δ must contain the whole curve. Hence ∞^2 surfaces of $|\bar{f}|$ contain δ. The images of these surfaces are ∞^2 planes having the image of δ in common, hence it must be a point. The curve must lie entirely on K, the surface of coincident points of (y), and the image point is a double point of L. (Segre (89).) If δ does not pass through O_y, this double point on L can not be in the characteristic singular plane of L.

(C) *If (y) contains a curve δ meeting $|f|$ only in basis points,*

and not passing through the basis point O_y of $|\bar{f}|$, L contains a double point not lying in its characteristic singular plane.

10. *Rational Plane Sections.* I. When the plane sections of a surface $f(y)=0$ are rational the system $|f|$ consists of ruled surfaces or of Steiner surfaces, that is, quartics with three concurrent double lines. (Picard (85), Guccia (63).) The simplest case satisfying conditions (a), (b), (c) is that of quadrics through 5 points. (Sharpe and Snyder (100).) The surface $L_4=0$ is now the 16-nodal Kummer surface, hence $V_3(x)=0$ is nodal. In case of surfaces of order greater than 2, the variable curve of intersection is always rational since the double curves of each f_i must belong to the basis curve of the system. (Castelnuovo (12), Enriques (30).)

11. *Plane Sections of Genus* 1. II. The simplest case is $N=3$. Conditions (a), (b), (c) are satisfied; in every case $L_4=0$ has a number of extra nodes. (Sharpe and Snyder (100).) Compare the general properties of linear systems of plane curves of general genus. (Segre (88).)

$N=4$ gives a system of quartic surfaces with a double conic and certain other basis elements. But this can be reduced to the preceding one by a quadratic transformation. For $N=5$ the basis elements are a double C_5 with a triple point. The residual intersection of any two surfaces of the system is an elliptic quintic curve; the system is ∞^6 of grade 5. The conditions (a), (b), (c) will be satisfied only by imposing two additional points; hence L_4 will have another singular plane and V_3^3 has nodes.

For N greater than 5, there is no residual curve. (Del Pezzo (26).) For varieties with curve sections of genus 1, see Scorza (86).

12. *Plane Sections of Genus $p=2$.* III. This will be discussed in connection with general hyperelliptic sections. The surface defined by $|C_6|:8P^2$ is the double quadric cone, which can be mapped on a quartic surface with a double line, which is associated with a nodal variety.

13. *Plane Sections of Genus* $p=3$. IV. The non-hyperelliptic case has been discussed by Castelnuovo (9) and by Scorza (87). Those reducible to quartics all have fundamental lines, hence they lead to nodal varieties. (Noether (83).)

Those of the first species, that is, those containing a net of rational curves of order 4, meeting by twos in a single point, can in general not be reduced to quartics.

The only exception is that of the monoids, represented on the plane by $C_4 : 12P^1$, the 12 basis points all lying on a cubic curve, which is the image of the triple point. Whatever system of basis curves is adjoined to obtain a subsystem satisfying conditions (a), (b), (c), fundamental lines are introduced, hence the associated V_3^3 has one or more double points. The rational quartic curves lie in the planes of the bundle having the triple point for vertex. For a discussion of the involutions which leave invariant a web of monoids, see Snyder (105).

The system $|C_4| \cdot 11P^1$ defines a rational quintic surface with a double cubic curve. All the possible forms of residual basis elements that can satisfy conditions (a), (b), (c) are given in (100); in each L_4, and hence V_3^3, has double points.

14. *Digression. Details of a Quintic Surface.* Case J, Sharpe and Snyder (100), merits further study. The table of characteristics as given is incomplete.

Using the notation there employed, the line h which is the partial image of h' in (x'), with which it is in (1, 1) correspondence, is double on each S_{21} of the web, since h' is double on each S_6' of the system in (x'). Moreover, the curves on H_2, which are the residual images of the points of h', are conics.

The curve β_5 meets H_2 in two points not on γ_3.

The line l joining them is axis of a pencil of planes, each of which cuts from H_2 a parasitic conic.

Two of these have for images in (x') the points of contact of h' and L_4', hence the conics lie on K_{10} and are double on

each S_{21} of the web. Each of the six lines $[R_4, K_{10}] = 6r$ also belongs to the base of $|S_{21}|$; hence we have

$$S_{21}: \quad \gamma_3^8\alpha^4\bar{\alpha}^4\beta_5^5 6r\rho_5 h^2 2c_2^2 ,$$
$$K_{10}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5^2 6r2c_2 .$$

A plane meets its conjugate S_{21} in a composite curve consisting of (S_1, K_{10}) and of a residual curve

$$\delta_{11}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5^3\rho_5 h^2 2c_2 .$$

Any two S_{21} may also have x_1 curves of order i, parasitic on the web $|S_5|$, and appearing to multiplicity i on $|S_{21}|$. The complete intersection then gives

$$441 = 21 + 192 + 16 + 16 + 125 + 6 + 5 + 4 + 16 + \sum i^2 x_i ,$$

and the intersection with K_{10} furnishes

$$210 = 10 + 96 + 8 + 8 + 50 + 6 + 8 + \sum i^2 x_i .$$

The only possible solution is $x_1 = 8$, $x_2 = 4$. The 8 lines are divided into two groups of four each, bisecants of β_5 meeting γ_3 and α or $\bar{\alpha}$. Hence we may now write

$$S_{21}: \quad \gamma_3^8\alpha^4\bar{\alpha}^4\beta_5^5 6r\rho_5 h^2 2c_2^2 8u_1 4u_2^2 ,$$
$$K_{10}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5 \; 6r2c_2 8u_1 4u_2 ,$$
$$\delta_{11}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5^3\rho_5 h^2 2c_2 4u_2 ,$$

where the interpretation of the symbols in the last equation undergoes an obvious modification.

The genus of δ_{11} is consequently 9, and it intersects K_{10} in 12 variable points. The images of these 12 points are the points of contact of δ' and L'; hence δ', the variable double curve on S_6', image in (x') of S_{21}, is of order 6. The genus of δ' is 2.

The curve δ_{11} has a double point D on h, and the plane of δ_{11} meets the image conic in two points P_1 and P_2. The points D and P_1 form one pair of conjugates, and D and P_2 another; hence δ_6' has a double point where it meets h', the tangents at which are distinct from each other and from h'. The point is a triple point on S_6'. The surface S_6' has a composite double curve consisting of δ_6' and of h', the latter passing through

a double point on the former. The genus of the composite c_7 can be found as follows. From an arbitrary point can be drawn 8 lines meeting δ_6' twice, accounting for 7 apparent double points, and the line to the double point. The plane determined by the point and h' meets δ_6' in 4 other points, hence $h=8+4=12$, $p=3$.

The web $|S_5|$ can be transformed birationally into a web of monoids. Let γ_3 be defined by

$$\frac{x_1}{x_2}=\frac{x_2}{x_3}=\frac{x_3}{x_4}.$$

Put $Q_1=x_1x_3-x_2^2$, $Q_2=x_2x_4-x_3^2$, $Q_3=x_1x_4-x_2x_3$. Consider the transformation defined by

$$y_i = Q_ix_4,\ (i=1,2,3),\ y_4 = \sum u_iQ_i,$$

where u_i is linear in x_1, x_2, x_3. The equations define a (3, 3) non-involutorial transformation between (x) and (y) in which the fundamental elements in (x) are γ_3 and a plane cubic $C_3: x_4=0, \sum u_iQ_i=0$ meeting γ_3 in three points. The image in (y) of the plane $x_4=0$ is the point $(0,0,0,1)\equiv O$. Every point of γ_3 is transformed into a straight line, the locus of which is a ruled surface R_5 of order 5, trisecants of an elliptic $C_6: O^3$. The image of a point of C_3 is a line through O, the locus of which is the cubic cone $\Gamma_3: C_6$.

The image of an arbitrary line of (x) is a cubic curve through O; a secant of γ_3 has for image a conic through O and a generator of R_5; a bisecant of γ_3 goes into a line through O and two generators of R_5. Thus the congruence of bisecants of γ_3 is transformed into the bundle O. Since every line of the congruence meets S_5 in one point not on γ_3, the image must be a monoid in (y). The complete image of S_5 is of order 15, but R_5^2 is a component; α^2, $\bar{\alpha}^2$ go into double basis lines through O.

The curve β_5 has for image a composite curve of order 15, consisting of a proper curve β_7 of order 7 and of eight lines on R_5. Since β_5 meets $x_4=0$ in 5 points, β_7 has O for five fold point. The residual $\bar{C}_6$ is simple,

$$s_6\colon O^4\alpha\bar{\alpha}\beta_7\bar{C}_6.$$

The variable curve of intersection is a C_{10} having O^4.

Since the variable C_6 meets each α in two points, the image C_{10} meets each image $\bar{\alpha}$ in two points, hence the curve lies on a quartic cone of genus 1, with two fixed double generators, $[\beta_7, \bar{C}_6]=6$.

15. *Resumption of the Case* $p=3$. This completes the enumeration of webs of rational surfaces with plane sections of genus 3, belonging to the first species. (Castelnuovo (9).) The normal surfaces are of order $16-\kappa$, lying in space of $14-\kappa$ dimensions ($\kappa<12$). Those that are not normal can be projected into one or another of the types just considered. (Clebsch (19), Cremona (24), Sturm (108).)

The F_6 with a double curve of order 7, genus 3, has an actual triple point on both curve and surface. (Bordiga (7), Veronese (110).)

The system $|C_4|:10P^1$ defines a sextic surface having a double curve C_7, $p=3$ with a triple point. The residual curve of order 8 is rational, hence condition (a) is not satisfied. Similarly for surfaces defined by $|C_4|$ having fewer than 10 basis points.

The second species is composed of those surfaces with plane sections of genus 3 which also contain an ∞^2 system of curves of genus 1; this system can be reduced to quartics. (Castelnuovo (9).) If the system of elliptic quartics is composed of plane curves, their planes pass through a point, and the surfaces are quartics having the given point for tacnode.

In every case the surface contains an $\infty^2 I_2$ of points, the lines joining pairs of conjugates being concurrent. From this point the surface can always be projected into a surface of Veronese in S_5, counted twice. The surface is always rational except when the arithmetic genus is -1 and when the curve of invariant points of the g_2^1 from O consists of four concurrent conics. These cases do not concern us. If the curves of genus 1 lying on the surface are space quartics,

the surface can be reduced to a quintic of S_3 having three concurrent double lines, and hence a triple point. The net of quartics is cut from the surface by the net of quadric cones passing simply through the double lines.

The F_8 of S_6, represented by $|C_6|:7P^2$, can be projected into an F_4 of S_3 by choosing the four centers of projection on the same elliptic curve lying on it. The result is a quartic of S_3 with a tacnode and four simple concurrent lines in the tangent plane at the tacnode. (Noether (80), Cremona (24).)

The system of curves $|C_6|:7P^2 3P^1$ defines an $F_5:3C_1^2 3C_1^1$, the three double lines being concurrent, and the three simple lines lying in the planes containing a pair of double lines. In the plane representation, let the double points be A, B, C, D, E, F, G and the simple ones P, Q, R.

The plane images of the three double lines are $C_3: A \cdots GPQ$; $C_3:A \cdots GPR$; $C_3:A \cdots GQR$. From the complete intersection of two surfaces of the system (F_5) the three double lines should be subtracted, hence the plane sections are represented by $|C_{12}|:A^4 \cdots G^4PQR$. Thus this characteristic is of grade 29, genus 13, dimension 17. Hence there are nineteen linearly independent surfaces of the system.

In order to secure a subsystem satisfying conditions (a), (b), (c), let (F_5) have an additional basis curve C_m, of order m, genus p, meeting the totality of the double lines in S points; then

$$11m - 2p + 2 - 5s = 26,$$
$$5m - p + 1 - 2s = 14,$$

so that $s=m+2$, $p=3m-17$. (Sharpe and Snyder (100), p. 71.) The residual intersection of genus 1 is met by any third surface in three points, hence it can be reduced to a cubic through six basis points. It has one of the four forms

(α) C_3: $A\,B\,C\,D\,E\,F$,

(β) C_3: $A\,B\,C\,D\,E\,P$,

(γ) C_3: $A\,B\,C\,D\,P\,Q$,

(δ) C_3: $A\,B\,C\,P\,Q\,R$.

In (α), the residual basis curve C_m is

$$C_9: \quad A^3 \cdots F^3G^4PQR.$$

The corresponding basis curve on the surfaces $|f(y)|$ is C_7, $p=4$, meeting each double line in 3 points. Through O can be drawn two other bisecants of C_7; these lines are fundamental, having five fixed points of intersection with each $f(y)$ of the system. Hence $L_4(x)$ has additional double points, and consequently V has a double point.

(β). The basis curve is now of order 6, genus 1, meeting two of the double lines each in 3 points and the third in 2 points. Hence as before, two additional bisecants can be drawn from O; these lines are fundamental, giving rise to double points on $L_4(x)$, and hence on V.

(γ). The basis curve is now composite, consisting of two conics which do not intersect, each meeting all the double lines once, and of a line skew to both conics, meeting one double line. The additional secant of both conics from O accounts for double points on V.

(δ). This case is impossible, as the residual basis curve is of order 4 and genus -5.

There is one more possible case of systems $|f(y)|$ where plane sections of genus 3 exists, namely, that having a plane representation of the form $|C_7|$; $7P^22P_1$. (Caporali (8).)

The double basis curve of the system $|f(y)|$ consists of a line C_1 and of a sextic γ_6, $p=2$, having two double points on the line. One surface of the system $|f(y)|$ must be composite, hence L_4 and also V_3 have double points.

The cases of systems of species 3 and of species 4, Castelnuovo (9), reduce to systems of quartics with point singularities more complicated than in that of type 2, but without basis lines. (Noether (83).)

The first surface is represented by $|C_6|$:$7P^24P^1$, all 11 basis points lying on a cubic curve γ. Let the singular point be (0, 0, 0, 1), and the tangent plane be $x_1=0$. Two surfaces of the system meet in a curve C_{16} of order 16, with 8 branches through O, all lying in $x_1=0$. The image of a plane section

through O consists of γ and of another cubic with the symbol C_3: $ABCDEFG$. The plane representation of C_{16} is $C_{12}:7P^4$. When this is composite, one component being of genus 1 and variable, it can be reduced to the $C_3:A \cdots F$, and the fixed basis curve has for image $C_9:A^3 \cdots F^3G^4$, $p=4$. In space the variable curve is a C_6 with three branches through O; the basis curve is then C_{10} with five branches through O. Since two trisecants of C_{10} can be drawn through O, these lines are fundamental, giving rise to double points on V_3^3.

It was assumed that the variable cubic in the plane representation had for basis points A, B, C, D, E, F; if instead one or more be chosen from the simple basis points of the system, practically the same argument may be repeated, with the same conclusion.

Another case is that of F_4 represented by $|C_7|:P^39P^2$, all the basis points lying on a cubic curve, and another is the F_4 represented by $|C_8|:8P^3P^2P^1$, the ten basis points lying on a cubic. Both of these can be considered as the preceding case, with the same result.

This completes the case $p=3$, non-hyperelliptic. When the net of curves on each surface of the system is composite, the plane sections of genus 3 are hyperelliptic. In this case the F_{16} is represented by $|C_6|:P^4P^2$ or by $|C_{8-k}|:P^{6-k}(3-k)P^2$, $(k=0, 1, 2, 3)$, the double points being adjacent to the P^{6-k}. They will be considered in connection with the general hyperelliptic case. (De Franchis (25).)

16. *General Hyperelliptic Sections.* V. From Castelnuovo (10) and Enriques (31) we know that every surface with hyperelliptic plane sections or by sections S_{n-1}, if it lies in S_n, is rational. If the genus of the sections exceeds 1, the surface contains a rational ∞^1 system of conics, such that through any point of the surface passes just one conic of the system. If the section is of order n and genus π, the surface cannot belong to a space larger than $S_{n-\pi+1}$; if it belongs to a space of lower dimensionality, it is the projection of a normal surface of $S_{n-\pi+1}$.

The planes of the ∞^1 conics on the surface form a variety of order not larger than $n-\pi-1$.

The surface can be mapped on a plane such that the plane sections (or those of S_{r-1} if in S_r) are represented by curves of order k having a common point of multiplicity $k-2$, and possibly other basis points. Hence if the genus is π, the maximum order is $4\pi+4$. Every such surface of lower order can be obtained from a surface of order $4\pi+4$ of $S_{3\pi+5}$ from $4\pi+4-n$ points upon it, and possibly from other points not on the surface.

In order to obtain a subsystem satisfying conditions (a), (b), (c), it is necessary to impose simple basis points in every case. When double points exist among the additional basis points, the curves can be reduced to those of lower order. Several cases of $|C_k|:P^{k-2}$ have been considered in connection with other problems. Let x be the number of simple basis points, N the order of the surface, m the order of its double curve. Then for (k, x, N, m) (4, 8, 4, 1) see (100), p. 61; (4, 7, 5, 3) see (100), p. 72; (4, 6, 6, 8) see (8), p. 202; (5, 11, 5, 3) see (100), p. 71; (5, 10, 6, 7) see (8), p. 207.

The system $|F_N|:C_1^{N-2}\ \gamma$ can be treated as in (100) p. 62, and the monoidal cases of any order necessarily have fundamental lines.

If the assumption made in § 8 is justifiable it follows that the general cubic variety of S_4 is irrational.

17. *Reduction to Monoidal Types.* From the preceding representation, follows at once the proof of the following theorem.

All (1, 2) *quaternary correspondences of genus* 1, *in which the quartic surface of branch points has a singular plane and at least one double point not in this plane, can be expressed in terms of a* (1, 1) *correspondence between the points of an* S_3' *and the pairs of points of a perspective monoidal involution of order two in another space* S_3.

Let there be a second involution I_2 in (x), having the same surface of branch points L_4' in (x'). Since the points of V_3

and of (x) are in (1, 1) correspondence, we have in any case $y_i = f_i(x_1, x_2, x_3, x_4)$, $i = 1, 2, 3, 4$. For example, by including the vertex P on V_3 from which the variety is projected on the double S_3, we may write

$$y_i = F_2(x)x_i; \quad y_5 = F_3(x).$$

Let the pairs of points of I_2 also be represented on V_3. Since the two surfaces of coincident points K_1 of I_1 and K_2 of I_2 are each in (1, 1) correspondence with L_4', they can be superposed one on the other.

Then I_2 is transformed into a second monoidal involution, having a second double point Q_x on K_2 for vertex. The image on V_3 of Q_x is a second double point, hence L_4' has a second double point. The line joining the double points on V_3 lies entirely upon it; every plane through it meets V_3 in a residual conic, image of a line of the bundle Q_x in (x). This line contains an infinite number of pairs of conjugate points with double points on K_x. Hence the conic on V_3 contains an infinite number of pairs of points forming an involution in which the double points are the residual points of intersection of the plane of the conic with the curve of contact K_V of the tangent cone from P. Thus, by I_2^V every point of K_V remains invariant; by projection upon $x_5' = 0$, the associated point on L_4' is a branch point as before.

If in (x) there exist several involutions having the same quartic surface of branch points with a singular plane, by the above process they can all be reduced to perspective monoidal involutions by the same transformation.

When L_4' has only one double point not in the singular plane, the web of invariant cubic surfaces in (x) has a simple point and a general C_6, $p = 4$, for basis elements. It is described as case D of cubics in Sharpe and Snyder (100), p. 58. It does not appear again in a transformed form in the enumeration there given.

If L_4' has two double points not in the singular plane, $K(x)$ has a second double point, the quadric F_2 containing C_6 and

the cubics of the web touch each other, C_6 has an actual double point and is now of genus 3.

Similarly, L_4' may have 3 or 4, . . . or 10 double points not in its singular plane. In the latter case it is the 16-nodal Kummer surface, and K may be reduced to the symmetroid.

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CORNELL UNIVERSITY

ON THE LOCATION OF ROOTS OF POLYNOMIALS AND ENTIRE FUNCTIONS*

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On account of its importance, the problem of the location of the roots of a polynomial occupies the central position in the theory of equations. For the most part the attention of mathematicians has been confined to the real roots of real polynomials—i.e., polynomials with real coefficients. So general has been the assumption of the reality of the polynomial that it has often, unfortunately, not been stated in theorems in which it is vital; for example, when it is said that imaginary roots enter in conjugate pairs. Correspondingly, the tools for the investigations were ordinary algebra and the calculus. When, however, one's thought is extended to the location of imaginary roots and to unreal polynomials, the border line is crossed, as the treatment involves almost inevitably considerations and methods belonging to the theory of analytic functions. It is in the overlapping region belonging alike to the theory of equations and that of analytic functions that we will stray today.

After the notable advances of Budan, Fourier, Cauchy, Laguerre and others, investigation of the root-location of polynomials and entire functions was largely neglected for other work. With the new century has come a new activity. It is my pleasure to bring before you today some of the important advances made and some theorems discovered since 1900. This will, I hope, be the more profitable because the new results lie scattered and untied through the periodical literature and have not yet become a part of current texts.† Some of them, though little known today, are destined to become classic.

* An address presented at the invitation of the program committee at a Symposium held at the meeting of the Society in Chicago, March 29, 1929.

† Except Bieberbach's noteworthy *Vorlesungen über Algebra*, 1928.

Time limitations force me to omit certain aspects and developments which I had purposed originally to include. Reluctantly I omit inclusion of the line of work* inaugurated by Landau and Montel. The underlying thought in this work is to give $p+1$ coefficients in an equation

$$(1) \quad P_n(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0, \quad (a_0 \neq 0),$$

leaving the other coefficients arbitrary, and to seek the limitations thereby imposed upon the roots. Thus Montel supposes the first $p+1$ consecutive coefficients $a_0, \cdots, a_p$ to be given in this equation or, more generally, in one of any degree with $n+1$ non-zero terms, and shows that then the p roots of smallest absolute value are conditioned to lie in a circle whose radius depends only on the values of these given coefficients and on n. I omit also consideration of special kinds of polynomials, computational methods for calculating the roots, polynomials with a parameter, chains of polynomials arising through continued fractions or other algorithms, extensions of Descartes' Rule of Signs or of the Hermite-Poulain Theorem, etc. I have not sought to give today an encyclopaedic report but rather so to select the material as to secure some unity of theme and thought.

At the end of the report is appended a bibliography which includes also some omitted topics allied to those considered here. Literature references to the bibliography are indicated in the report by bracketed numbers.

PART I

One succession of modern theorems was inaugurated by the Gauss-Lucas Theorem on the location of the roots of the derivative of any polynomial, real or imaginary. This theorem, though so fundamental, is strangely enough but little known popularly.

GAUSS-LUCAS THEOREM. *If in the complex plane of $z=x+iy$ we mark off the convex polygon whose vertices are*

* See the bibliography.

roots of a given polynomial and which contains either within or upon its boundary all other roots of the polynomial, then none of the roots of the derived polynomial will lie exterior to this polygon, and a root will fall upon its boundary only when coincident with a multiple root of the given polynomial on the boundary.

This theorem was definitely stated and proved by Lucas [63] in 1879 and perhaps can be best known by his name, but it follows as an immediate corollary from an observation of Gauss [62] appended as a *handschriftliche Bemerkung* after a memoir in his *Collected Works*. Gauss says that the roots of the derivative of any given polynomial are the positions of equilibrium in a plane field of force created by placing unit-masses at the roots of the given polynomial which repel (or just as well attract) with a force inversely proportional to the distance. A mass r is, of course, to be placed at each r-fold root. For proof of this Equilibrium Theorem of Gauss* we have merely to observe that the roots of the derivative are given by the equation

$$(2) \qquad \frac{P_n'(z)}{P_n(z)} \equiv \sum \frac{1}{z-\alpha_j} \equiv \sum \frac{1}{r_j e^{i\theta_j}} = 0, \qquad (j = 1, \cdots, n),$$

where the α_j are the roots of the given polynomial and the r_j, θ_j denote the lengths and arguments of the vectors drawn from these roots to a point z. Then, if each θ_j is changed into $-\theta_j$ to convert the terms into their conjugates, each term in (2) becomes a vector representing a force due to a unit mass situated at α_j which repels inversely as the distance, and the equation gives the positions of equilibrium, where there is no force. Now it is obvious that if we produce any side of the convex polygon indefinitely and z is a point within the half plane not containing the polygon, the components of force at z perpendicular to the side will all be directed away from the polygon, thereby reinforcing each other so that equilibrium at z is impossible. Like considera-

* Rediscovered by Lucas, Comptes Rendus, vol. 89 (1879), pp. 224–226.

tions apply when z lies on a side and is not coincident with a root of the given polynomial. The theorem of Lucas thus follows as a corollary.

Beside this theorem of Gauss-Lucas we may now place a beautiful result published without proof by Jensen [72] in 1913, which gives much more definite information regarding the derived polynomial when the given polynomial is *real.*

JENSEN'S THEOREM. *If circles are described having as diameters the segments connecting pairs of conjugate roots of the given polynomial, then no imaginary root of the derivative of the polynomial can lie exterior to all these circles.*

The first published proof [73] of this theorem we owe to Walsh (1920–1921); it is likewise based on the above Equilibrium Theorem of Gauss. Walsh sets a repulsive unit-mass at each root of the given polynomial and shows that at any point not on the real axis and exterior to the circle having as diameter the segment joining a pair of conjugate imaginary roots α_j, $\bar{\alpha}_j$, the net force due to the masses at these two points will have a non-vanishing component perpendicular to the real axis and directed away from this axis. The like is obviously true of the repulsive force due to the masses upon the real axis. Hence at any point exterior to all the Jensen circles and not on the real axis these components reinforce one another and equilibrium is impossible. Consequently no imaginary root of the derived polynomial can lie exterior to all the Jensen circles. Walsh apparently did not notice that the introduction of repulsive masses was an unnecessary trapping and that without essential modification the proof could be thrown into the following elementary form [74], quite possibly that of Jensen himself. Let $\alpha=\beta+i\gamma$, $\bar{\alpha}=\beta-i\gamma$ be a pair of conjugate imaginary roots of $P_n(z)$. The corresponding component in (2) is

$$\frac{1}{(x+iy)-(\beta+i\gamma)}+\frac{1}{(x+iy)-(\beta-i\gamma)}$$
$$=\frac{2(x-\beta)[(x-\beta)^2+y^2+\gamma^2]-2iy[(x-\beta)^2+y^2-\gamma^2]}{[(x-\beta)^2+(y-\gamma)^2]\cdot[(x-\beta)^2+(y+\gamma)^2]}.$$

The coefficient of i is opposite in sign to y if $(x-\beta)^2+y^2-\gamma^2>0$; that is, when z lies outside the circle $(x-\beta)^2+y^2=\gamma^2$. For a real root β the component is

$$\frac{1}{x+iy-\beta}=\frac{(x-\beta)-iy}{(x-\beta)^2+y^2},$$

which also has a sign opposite to that of y for the coefficient of i. Hence at any point exterior to all the Jensen circles and not on the real axis the coefficient of i in P_n'/P_n does not vanish. This proves the theorem.

Walsh supplements Jensen's Theorem by showing that in any segment of the real axis which is free from roots of the real polynomial $P_n(z)$ and which does not penetrate into any of its Jensen circles there can lie at most only one root of its derivative $P_n'(z)$. Further, if any Jensen circle has no point in common with any other Jensen circle in part exterior to it and contains exactly k roots of $P_n(z)$, then it will contain at least $k-1$ roots of its derivative and at most, $k+1$ roots. I may add that I find that if we amalgamate each series of overlapping circles into a single connected region so as to form Jensen regions exterior to one another, the conclusion still holds with respect to these regions.

These results can be readily established analytically without following the Gaussian suggestion of introducing repulsive masses at the roots of the given polynomial, but that method is undeniably illuminating and suggestive, and has the advantage of being independent of the selection of the coordinate axes. Bôcher [68] applied the method to the Jacobian

$$\begin{vmatrix} \dfrac{\partial\phi}{\partial z_1} & \dfrac{\partial\phi}{\partial z_2} \\ \dfrac{\partial\psi}{\partial z_1} & \dfrac{\partial\psi}{\partial z_2} \end{vmatrix}$$

of two binary forms $\phi_{p_1}(z_1, z_2)$, $\psi_{p_2}(z_1, z_2)$ of degrees p_1 and p_2 respectively. It is easy then to see that the roots $z=z_1/z_2$

of the Jacobian, other than multiple roots of either form or common roots of both forms, are the equilibrium positions in a field of force due to unit masses placed at the roots of ϕ and of ψ which respectively attract and repel with a force inversely proportional to the distance. Continuing the application, Walsh has made an extended study of the location of the roots of the Jacobian. In a number of memoirs he has also applied the Gaussian idea to derive some nice theorems relating to the roots of the derivative of a polynomial. For example, he shows [71] that if m_1 roots of a polynomial lie in a circle of radius r_1 and center α_1 and all the remaining m_2 roots lie in a circle of radius r_2 and center α_2, all the roots of the derivative lie in these two circles and a third circle whose center is $(m_1\alpha_2+m_2\alpha_1)/(m_1+m_2)$ and whose radius is $(m_1r_2+m_2r_1)/(m_1+m_2)$. If, furthermore, these circles are all external to one another, then the three circles contain respectively m_1-1, m_2-1, and one root respectively.

I may add that the Gaussian method and various resulting conclusions—for example, Lucas and Jensen's theorems—apply not only to the derivative of the given polynomial $P_n(z)$ but also to any polynomial $\psi_{n-1}(z)$ of degree $n-1$*, for which we have a decomposition into partial fractions of the form

$$\frac{\psi_{n-1}(z)}{P_n(z)} = \sum_{j=1}^{n} \frac{A_j}{z-\alpha_j},$$

in which the A_j are all positive.

The next theorem which I wish to cite relative to the roots of the derivative of a polynomial is due to Grace and arose from an attempt to obtain a generalization of Rolle's Theorem valid for two unreal roots of a polynomial. According to Rolle's Theorem, if a *real* polynomial vanishes at two different points of the real axis (or has the same value at these points), there will lie at least one root of its derivative between them. Thus two real roots of a real polynomial *fixate*

* For extension of the Jensen Theorem, the polynomial $\psi_{n-1}(z)$ is, of course, to be taken real as well as $P_n(z)$.

one root of the derivative. In what direction should one look for a corresponding result for imaginary roots and unreal polynomials? Doubtless this question occurred to others before Grace, and we may conjecture that it was on account of its difficulty that no answer was obtained till the present century. In November 1901, in a notable five-page article [75], Grace obtained the following important and strange theorem.

THEOREM OF GRACE. *If a and b are any two roots of any polynomial, real or unreal, of degree n, then within or on a circle having the center $(a+b)/2$ and a radius one-half of $|a-b|$ ctn (π/n) there will be at least one root of the derived polynomial.*

This can scarcely be called a generalization of Rolle's Theorem inasmuch as it does not reduce to Rolle's Theorem for the special case of a real polynomial with two real roots. Further, in anticipation of the second part of my report today, I may add that it is not a theorem that admits of extension to entire functions, since as n increases indefinitely, the radius of the circle also increases indefinitely. No information could therefore be afforded regarding the derivative of the given entire function except that it had a root somewhere in the complex plane. But this conclusion would be false since the entire function e^z-1 has a periodic system of zeros, while its derivative is devoid of zeros.

Grace's proof of his theorem was essentially invariantive in character. If in an algebraic equation

$$(1) \qquad P_n(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0,$$

we put $z = x_1/x_2$, the left-hand member after dropping the denominator is converted into an algebraic form in x_1 and x_2. If we take a second equation

$$(3) \qquad b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n = 0,$$

the two equations may be said to be *apolar* when the corresponding bilinear forms are apolar; that is, if their coefficients are connected by the equation

$$(4)\quad a_0b_n - \frac{1}{C_1^n}a_1b_{n-1} + \frac{1}{C_2^n}a_2b_{n-2} - \cdots + (-1)^n a_nb_0 = 0.$$

Starting from Lucas' Theorem, Grace first establishes between two apolar equations a most important and far reaching relation which I shall term Grace's Apolarity Theorem.

GRACE'S APOLARITY THEOREM. *If two algebraic equations are apolar, then any circle in the z-plane which contains all the roots of either equation will contain at least one root of the other.*

The term circle is here to be understood in the sense of the theory of functions and may be indifferently the closed interior or the closed exterior of a circle, or a half plane inclusive of its boundary.

It will be noticed that if any two of the equations (1), (3), (4) are given, we can immediately write down the third. Hence the theorem says that if any algebraic equation (1) is given whose coefficients are connected by a homogeneous linear relation (4), we can immediately write down a third equation (3) such that every circle which contains all the roots of (1) will contain at least one root of (3), and reciprocally.

Grace applies this thorem to the derivative of (1). If we substitute in (1) any two of its roots α and β and subtract, we obtain a homogeneous linear relation

$$a_1(\alpha - \beta) + a_2(\alpha^2 - \beta^2) + \cdots + a_n(\alpha^n - \beta^n) = 0$$

between the coefficients of the derived equation. The equation (3) which corresponds to this linear relation is

$$(\alpha^n - \beta^n) - n(\alpha^{n-1} - \beta^{n-1})z + C_2^n(\alpha^{n-2} - \beta^{n-2})z^2 + \cdots + (-1)^{n-1}C_1^n(\alpha - \beta)z^{n-1} = 0,$$

which may be written

$$(5)\qquad (\alpha - z)^n - (\beta - z)^n = 0.$$

Hence at least one root of the derivative $P_n'(z)$ must lie in

any circle containing all the roots of (5). By extracting the nth root in (5) we get for its roots

$$z = \frac{\beta\omega_j - \alpha}{\omega_j - 1}, \qquad (j = 1,2, \cdots, n-1)$$

where $\omega_j \equiv \cos(2j\pi/n) + i \sin(2j\pi/n)$ denotes an nth root of unity. This can be easily reduced to the form

$$z = \frac{\alpha+\beta}{2} + i\frac{\alpha-\beta}{2} \operatorname{ctn}\left(\frac{j\pi}{n}\right),$$

which shows that the n roots of (5) all lie on a straight line perpendicular to the segment joining α and β at its middle point. The most remote of these roots is at a distance one-half of $|\alpha-\beta|$ ctn (π/n) from this middle point. Consequently a circle described around this mid-point as center and with a radius equal to this distance contains all the roots of the equation (5) apolar to the derivative $P_n'(z)$ and hence by Grace's apolarity theorem encloses at least one root of $P_n'(z)$. Thus two roots α and β of a given polynomial fixate one root of the derivative in the manner stated.

In 1907 this theorem was re-derived, apparently independently, by Heawood [76]. He noted the fact that for $n>3$ the region in which lies at least one root of the derivative need not be extended equally in every direction from the center $(\alpha+\beta)/2$ to the perimeter of the circle. When $n=4$, the minimum region is that covered by two circles of radius one-third of $|\alpha-\beta|\operatorname{ctn}(\pi/n)$ and having their centers on the perpendicular to the segment joining α and β at its middle point and at a distance one-sixth of $|\alpha-\beta|\operatorname{ctn}(\pi/n)$. For greater values of n the form of the curve bounding the region is very complicated and scarcely specifiable. Heawood established Grace's Theorem by showing that the root of the derivative nearest to the mid-point $(\alpha+\beta)/2$ attains its maximum distance only when all the roots of the derivative are equal; that is, only when $P_n'(z) \equiv c(z-\gamma)^{n-1}$, so that we may write $P_n(z) \equiv c(z-\gamma)^n/n + c_1$. Substituting the roots α

and β in turn, we find $(\alpha-\gamma)^n \equiv \beta-\gamma)^n$, and γ is consequently a root of equation (5). The theorem then follows as before.

The most interesting successor of Grace's memoir is one published in 1922 by Szegö [77] with the modest title: *Bemerkungen zu einem Satz von J. H. Grace ueber die Wurzeln algebraischer Gleichungen.* He derives Grace's Theorem on apolar polynomials on the basis of what he calls a *Faltungssatz*, which I shall call his *Coincidence-Theorem.* This runs as follows.

SZEGÖ'S COINCIDENCE THEOREM. *If S_1 denotes the sum of the products of $z_1, z_2, \cdots, z_n$ taken j at a time and these are any n quantities satisfying the equation*

$$a_0 + a_1 S_1 + \cdots + a_n S_n = 0, \tag{6}$$

then any circle which contains all the roots of the equation

$$a_0 + C_1^n a_1 z + C_2^n a_2 z^2 + \cdots + a_n z^n = 0 \tag{7}$$

will contain at least one of any of the n quantities satisfying (6).

It will be noticed that equation (7) results from (6) when the z_j are all coincident. Szegö's proof, not altogether easy, is by means of mathematical induction. An obvious consequence of his theorem is that if any set of the z_j satisfying (6) all lie *in* (that is, on or within) a circle, not all the roots of (7) lie without. We have thus Walsh's theorem.

THEOREM OF WALSH. *If any circle contains a set of values of $z_1, \cdots, z_n$ satisfying* (6), *it must contain at least one root of* (7).

This result, in a somewhat different formulation, was published almost simultaneously as Theorem I and the "main result" of one of Walsh's principal papers [78] and was established by geometrical considerations involving continuity. It is not, however, to be regarded as a subordinate corollary of Szegö's Theorem, inasmuch as the latter is obviously also an immediate corollary of the former. The relation of the two theorems was not pointed out by Walsh.

The usefulness of the Szegö-Walsh Theorem is revealed

by very different sets of theorems which the two authors rapidly deduce from it. I have time to mention only one or two of each. As a typical theorem of Walsh, I select the following [78, p. 173]: Let

$$(8) \qquad (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)-A=0$$

be an equation with a parameter A; then if $\alpha_1, \cdots, \alpha_n$ are all situated in a circle of radius r and center α, the n roots of (8) will all be contained in n circles whose centers $\alpha+A^{1/n}$ form a regular polygon with center at α and which have the common radius r. If none of these circles overlap, there will be just one root in each circle. The second of his theorems which I wish to cite is a special case of his "Main result" and is set off as his Theorem II.* This says that "if the points $\alpha_1, \alpha_2, \cdots, \alpha_k$ lie in a circular region C and if z is any point exterior to C, the root of the equation in α

$$(9) \qquad \frac{1}{z-\alpha_1}+\frac{1}{z-\alpha_2}+\cdots+\frac{1}{z-\alpha_k}=\frac{k}{z-\alpha}$$

lies in C." Rephrased in the language of Gaussian attractions, it asserts that when unit masses are placed at k points $\alpha_1, \cdots, \alpha_k$ lying in a circle and the entire mass k is then concentrated at a single point α so as to produce the same resultant force at a given exterior point z, the point α must lie somewhere in the circle.

I have cited this second theorem of Walsh because, in considering a very different matter, I have been led to the same equation but with z as the unknown in place of α. If a polynomial equation $P_n(z)=0$ and its first derived equation are both transformed by the same entire linear substitution $z=az'+b$, they pass over into a new equation and its first derived equation. This is not the case when a fractional linear substitution

$$(10) \qquad z=\frac{az'+b}{cz'+d}, \qquad (c\neq 0),$$

* Curtiss [78a] points out that the theorem is a "result which Laguerre expressed in homogeneous coordinates."

is applied. Nevertheless, various theorems regarding the location of the roots of a polynomial and its derivative may continue to hold in the transformed region with respect to the transformed polynomial and its derivative. Consider, for instance, a theorem less specific than the above Gauss-Lucas Theorem, according to which a circle or half-plane which contains all the roots of $P_n(z)$ also contains all the roots of its derivative. Apply now the transformation (10). In place of $P_n(z)$ we get the new polynomial

$$Q_n(z') \equiv (cz' + d)^n P_n\left(\frac{az' + b}{cz' + d}\right). \tag{11}$$

Suppose that the transformation converts the interior of the circle or half-plane into the interior of a new circle C'. Then the new circle which necessarily contains all the roots of $Q_n(z)$ will contain also all the roots of $Q_n'(z)$. The like conclusion fails, however, when the transformed region is the exterior of C'. Similarly, if Jensen's Theorem were not invalidated by a transformation converting the half-plane into the interior of C', we would have the following conclusion: When the roots of a polynomial $Q_n(z)$ which do not lie on the perimeter of a given circle C' are pairs of roots of equal multiplicity symmetrically located* with respect to C', all the roots of its derivative not on the circle must lie in the circles whose diameters are the segments joining the pairs of symmetrically situated roots. It is easy to show the falsity of such a conclusion by an example to the contrary. Jensen's Theorem, however, is an immediate result of two applications of the following true statement: The roots of a real polynomial which do not lie in a half-plane bounded by a real axis lie in the Jensen circles. The corresponding statement after the above transformation is that when a polynomial $Q_n(z)$ has its roots located in the manner just explained, the roots of the derivative so far as they do not lie on or within

* Two symmetrically situated points are inverse points with respect to the circle.

C' must lie in the circles whose diameters are the segments joining pairs of symmetrically situated roots of $Q_n(z)$, and the validity of this statement has been established by Walsh through the method of repulsive particles.

These examples suffice to show the need of some criterion for determining whether a theorem regarding the mutual location of the roots of a polynomial and its derivative will hold also for the transformed polynomial and its derivative after a transformation (10). To answer this question, suppose $P_n(z)$ replaced by (11) in consequence of the transformation (10). The new derived equation is

$$Q_n'(z') \equiv nc(cz' + d)^{n-1}P_n\left(\frac{az' + b}{cz' + d}\right) + (ad - bc)\cdot(cz' + d)^{n-2}P_n'\left(\frac{az' + b}{cz' + d}\right) = 0,$$

which may be successively written

$$\frac{nc(cz' + d)}{ad - bc} + \frac{P_n'(z)}{P_n(z)} = 0,$$

and

$$(12) \qquad -\frac{n}{z - \frac{a}{c}} + \sum \frac{1}{z - \alpha_1} = 0, \qquad (i = 1, \cdots, n),$$

where the α_i are the roots of the original polynomial $P_n(z)$. It thus appears that the new derived equation $Q_n'(z') = 0$ results by application of the linear transformation (10) to the polynomial equation (12). Now (12) is Walsh's equation (9) regarded as an equation of $(n-1)$th degree in z with given α_i and α. It may be written in the form

$$(12a) \qquad nP_n(z) - (z - \alpha)P_n'(z) = 0,$$

where α denotes the point $z = a/c$ which by the transformation passes over into $z' = \infty$. For given α, there corresponds to each pair of polynomial equations

$$P_n(z) = 0, \quad Q_{n-1}(z) \equiv nP_n(z) - (z - \alpha)P'_n(z) = 0,$$

a class of substitutions (10) with $a/c=\alpha$ which replaces them by a polynomial equation $Q_n(z')=0$ and its first derived equation. Conversely, for the latter pair, there is a class of (inverse) substitutions which replaces them by pairs $P_n(z)=0$, $Q_{n-1}(z)=0$ with given α.

The question whether a theorem regarding the root location of $P_n(z)$ and its derivative is valid after the transformation for $Q_n(z)$ and its derivative therefore reduces to the determination whether the original theorem is valid for $P_n(z)$ and $Q_{n-1}(z)$ before the transformation as well as for $P_n(z)$ and $P_n'(z)$. It will be observed that α may have three qualitatively different positions with respect to the interior of a circle or half-plane. It may lie without, within, or on its boundary. If this interior is to be transformed by (10) into the interior of a new circle C', the point α must be taken without the circle C or the half-plane, since it passes over into $z'=\infty$.

So far as I know the above relationship has not been previously pointed out. I find, however, that Gonggrÿp [99] has considered the polynomial $nP_n(z)-zP_n'(z)$ which arises by application of $z=1/z'$ to $P_n'(z)$. He calls it the "*reduite*" of $P_n(z)$ and gives several theorems relating to its roots.

As an example of the general interrelation just pointed out, let us take Walsh's "Theorem 2" previously cited, which says that when z is exterior to a circle containing all the roots of $P_n(z)$, the point $\alpha=a/c$ in (9) must lie in the circle. The word circle has here the general sense of the closed interior or exterior. An equivalent statement is that when the roots of $P_n(z)$ are all in the circle and a/c lies without, the z-roots of (12) must all lie in the circle. By the transformation (10) this gives the restricted Lucas Theorem that when a circle contains all the roots of a polynomial $Q_n(z')$, it contains also the roots of its derivative. Conversely, Walsh's Theorem follows from this by the inverse linear transformation.

I have not time to follow the matter further but return now to the Coincidence Theorem of Szegö. It has been already

noted that he uses it to obtain Grace's Apolarity Theorem. His easy deduction is reversible, so that the two theorems, so very different in appearance, are equivalent. Szegö then goes on to obtain in an interesting manner a wealth of theorems. The only one of these which I shall cite is a theorem of Egerváry:

THEOREM OF EGERVÁRY. *Let*

$$A(x) \equiv a_0 + C_1^n a_1 z + C_2^n a_2 z^2 + \cdots + a_n z^n = 0, \tag{13}$$

$$B(x) \equiv b_0 + C_1^n b_1 z + C_2^n b_2 z^2 + \cdots + b_n z^n = 0 \tag{14}$$

be two equations whose roots all lie in (within) the unit circle. Then the roots of the composition-equation

$$C(x) = a_0 b_0 + C_1^n a_1 b_1 z + \cdots + a_n b_n z^n \tag{15}$$

all lie in (within) the same circle.

For proof [77], denote by $\beta_1, \beta_2, \cdots, \beta_n$ the roots of (14), and by ξ any root of (15), so that

$$a_0 b_0 + C_1^n a_1 b_1 \xi + C_2^n a_2 b_2 \xi^2 + \cdots + a_n b_n \xi^n = 0. \tag{16}$$

Then if we substitute $-\xi/z$ for z in (14), the resulting equation has the roots $-\xi/\beta_1, \cdots, -\xi/\beta_n$. The condition that this equation and (13) shall be apolar is precisely the equation (16). Then by Grace's apolarity theorem some one of these roots, say $-\xi/\beta_i$, must lie in the unit circle when all roots of (13) so do. This is impossible if ξ lies without (respectively without or on) the unit circle, since $|\beta_i| \leqq 1$ (respectively $|\beta_i| < 1$) by hypothesis. The theorem therefore follows.

If we restrict the hypothesis further and suppose that all roots of (13) and of (14) lie on the perimeter of the unit circle, then in applying the theorem just proved, we can take at pleasure the circle to be the interior or the exterior of the unit circle. It follows that no root of (15) can lie without, respectively within; consequently the roots of the composition equation will lie on the perimeter of the unit circle when those of (13) and (14) do.

Very similar in character are theorems given by Malo* (1895) and Schur (1914). Malo's theorem [88] says that if the roots of both of the equations

$$a_0 + a_1 z + \cdots + a_m z^m = 0, \tag{17}$$

$$b_0 + b_1 z + \cdots + b_n z^n = 0 \tag{18}$$

are all real and those of the former are all positive or all negative, then the roots of the composition equation

$$a_0 b_0 + a_1 b_1 z + \cdots + a_k b_k z^k = 0, \tag{18a}$$

in which k denotes the smaller of the two degrees m and n, are also real; furthermore, they are all different if $a_0 b_0 \neq 0$ and $n \leqq m$. Schur [90] draws the same conclusion for another composition equation:

$$a_0 b_0 + 1! a_1 b_1 z + 2! a_2 b_2 z^2 + \cdots + k! a_k b_k z^k = 0.$$

If now we look upon (17) as any arbitrarily given equation with real roots of like sign, we may regard $b_0, b_1, \cdots, b_n$ as a set of multipliers which by composition with (17) will always produce an equation (18a) with only real roots. Pólya and Schur [92] call any sequence of real multipliers with this property a factor sequence of the *second kind*. A sequence of real factors

$$b_0, b_1, b_2, \cdots \tag{19}$$

is said to be of the *first kind* when operating on all equations (17) with real roots ($n = 1, 2, \cdots$) they produce only composition equations (18a) with real roots. Laguerre in his memoirs gave various special sequences with such properties; for instance,

$$1, q, q^4, q^9, q^{16}, \cdots, \qquad |q| \leqq 1,$$

is a sequence of the first kind. Pólya and Schur make a most interesting study of the conditions which must be imposed upon the infinite set (19) of real numbers to furnish a se-

* Malo's theorem is extended by Jensen [72] to series.

quence of the first or second kind. If, for example, (19) is a sequence of the first kind, all non-zero elements must form a single uninterrupted succession, finite or infinite in length, the terms of which are all of same sign or of alternating sign. The two authors obtain both an algebraic and a transcendental necessary and sufficient criterion that it shall be a factor sequence of the first or second kind respectively. According to the algebraic criterion, (19) is a factor sequence of the first kind if and only if all roots of the equations

$$b_0 + C_1^n b_1 z + C_2^n b_2 z^2 + \cdots + C_n^n b_n z^n = 0, \quad (n = 1,2,3, \cdots),$$

are real and of like sign, while it is one of the second kind when they are merely real. By the transcendental criterion it is necessary and sufficient that the real series

$$b_0 + \frac{b_1}{1!}z + \frac{b_2}{2!}z^2 + \frac{b_3}{3!}z^3 + \cdots$$

should converge for all values of z and represent an entire function which has only roots of the same sign and is of genre 0 except for a possible factor $e^{\gamma z}$, in which γ has the sign opposite to that of the roots. For a factor sequence of the second kind this series must be of genre 0 or 1 except possibly for a factor $e^{-\gamma^2 z^2}$ (γ real) and have only real roots.

Somewhat earlier Petrovitch [91] considered factor sequences (19) having the property that when operating upon real polynomials of even degree possessing only imaginary roots they produce always polynomials without real roots. A class of sequences specified by him with this property has, when somewhat restricted, the form

$$b_n = \int_c^d f(t)t^n dt, \qquad (n = 1,2, \cdots),$$

where c and d are arbitrary fixed real limits independent of n and $f(t)$ is an arbitrary real function of invariable sign between c and d. For if for $m = 2n$ we denote the polynomial (17) by $P_{2m}(z)$, the composition equation will be

$$0 = a_0b_0 + a_1b_1z + \cdots + a_nb_nz^n \equiv \int_c^1 f(t)P_{2n}(tz)dt,$$

which can not be satisfied for any real value of z inasmuch as both factors of the integrand are of invariable sign. A special set of factors is also obtained by putting $b_n = n!$.

PART II

An infinite series $a_0+a_1z+a_2z^2+\cdots$ convergent over the entire finite plane of z is, in a certain sense, a generalization of a polynomial of the nth degree $a_0+a_1z+\cdots+a_nz^n$. We call it an *entire function*. The futility of efforts to carry over theorems regarding polynomials to entire functions without restriction upon the latter is, I suppose, well known to you. A familiar example is afforded by the Fundamental Theorem of Algebra. Though a polynomial of nth degree has n roots, an entire function need not have any zeros at all. The familiar exponential function $e^z = 1+z+z^2/2!+\cdots$ is an example of such an entire function. Again, take the familiar theorem that if a polynomial has only real roots, its derivative has only real roots. By contrast, $(z+1)e^{z^2}$ is a real entire function whose only zero $z=-1$ is real, but its derivative $(2z^2+2z+1)e^{z^2}$ has two imaginary zeros [132]. The question therefore arises: What sort of limitations must be imposed upon entire functions to enable us to extend to them our theorems on the location of roots of polynomials?

Investigation in this direction was inaugurated by Laguerre in the 80's or earlier. Since then little advance has been made until recently when some very interesting extensions have been obtained, notably by Pólya. Before reporting on these I must recall to you certain results and ideas familiar to the analyst. First of all, I must mention the theorem of Hurwitz that when a sequence of polynomials $P_n(z)$, $(n=1, 2, \cdots)$, or analytic functions converges uniformly over a closed region, the roots of the limiting analytic function within the region are the limits of the zeros of the polynomials in the sequence. In the vicinity of a p-fold

root of the limit lie exactly p roots of $P_n(z)$ for sufficiently large values of n. An interesting application of this indispensable theorem is to the exponential series: in any given finite region of the plane there can lie no roots of the sum $S_n(z)$ of the first n terms for sufficiently large values of n.

Next, I must mention Laguerre's concept of the *genre* of an entire function. According to Weierstrass's Factor-Theorem every entire function can be factored into the uniformly convergent form:

$$F(z) = e^{G(z)} z^r \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{H_n(z)}, \tag{20}$$

where

$$H_n(z) = \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n},$$

and where $G(z)$ is an entire function, polynomial or constant, r is a non-negative integer, and m_n is an appropriate positive integer which may increase indefinitely with n. When the degree m_n is limited and $G(z)$ is a polynomial, the function is said to be of finite genre, the genre p being then the highest (necessary) degree in the exponents of e within or without the product sign. The entire function (20) differs in structure from a polynomial by the presence of exponential factors. Hence, as Borel has remarked, one may expect that the properties of entire functions will depart more and more from those of polynomials as we increase the degree of the exponential factor.

For $p=0$ we have

$$F(z) = Cz^r \prod \left(1 - \frac{z}{a_n}\right),$$

so that

$$\frac{d \log F(z)}{dz} \equiv \frac{F'(z)}{F(z)} = \frac{r}{z} + \sum \frac{1}{z - a_n}, \tag{21}$$

where Π and Σ are uniformly convergent over any closed finite region of the plane not containing any roots a_n of the entire function. Correspondingly, we have for $p=1$,

$$F(z) = C_1 e^{cz} z^r \prod \left(1 - \frac{z}{a_n}\right) e^{z/a_n},$$

$$\frac{F'(z)}{F(z)} = c + \frac{r}{z} + \sum \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right); \tag{22}$$

and for $p=2$,

$$F(z) = C e^{c'z^2 + c''z} z^r \prod \left(1 - \frac{z}{a_n}\right) e^{z/a_n + (1/2)(z/a_n)^2},$$

$$\frac{F'(z)}{F(z)} = c'' + 2c'z + \frac{r}{z} + \sum \left(\frac{1}{z - a_n} + \frac{1}{a_n} + \frac{z}{a_n^2}\right). \tag{23}$$

When $F(z)$ is a function of finite genre and its Weierstrassian product (20) is devoid of the plaguy exponential factor before the product sign,—that is, when $G(z)$ is a constant—we call the function *primitive* or *canonical*.

We are ready now for the extension of polynomial theorems to entire functions. Consider first the simple theorem that when the roots of a polynomial are all real, the like is true of the derivative. The simple example already given shows that this may fail for an entire function of as low genre as 2. In the case of the real polynomial with only real roots we locate all the $n-1$ roots of the derivative by Rolle's Theorem which puts one between each two successive roots of the given polynomial, but the argument fails in the case of a real entire function because the derivative of an entire function may have an infinite number. In the case of real entire functions of genre 0 or 1 the result is still valid. For first, it can be established that there can be no imaginary roots of $F'(z)$, either by applying the equilibrium theorem of Gauss to the right-hand members of (21) and (22) or, what amounts to the same thing, by observing that the sign of the imaginary component of each term is opposite to y in $z=x+iy$, since c and the a_n are real, so that $F'(z)$ will not vanish unless $y=0$. Next, if we differentiate the right-hand members of (21) and (22) their terms will all be negative for a real value of z. Consequently $F'(x)/F(x)$ is a monotone

decreasing function of x, and can vanish but once between two successive roots of the denominator $F(x)$. Thus the roots of the derivative of a given real entire function of genre 0 or 1 are all real and alternate with those of the function itself, as in the case of a given real polynomial.

The argument just indicated fails in the case of entire functions of genre 2 because of the additional terms introduced in (23). When, however, c' is negative and Π is a canonical product of genre 1 or 0 so that the term $(z/a_n)^2$ is lacking in Σ, the argument carries through. We are led thus to pick out the same special class of functions of genre 2 which we ran across at the close of Part I, namely, real entire functions which are of genre 0 or 1, except for an exponential factor $e^{c'z^2}$ in which c' is negative.

I digress a moment at this point to mention a more general result that was just obtained. A real entire function of genre p with q imaginary roots possesses a derivative of the same genre which in addition to the roots accounted for by the theorem of Rolle can have at most $p+q$ imaginary and other real roots. A rigorous proof of this theorem of Laguerre is given by Borel [132].

I pass next to the consideration of another noteworthy case in which the derivative of the real entire function $F(z)$ possesses only real zeros which alternate with those of $F(z)$ on either side of the origin. It is that in which $F(z)$ is a canonical function, limited by a restriction placed upon it which at first sight appears very curious, viz., *that it shall have no zero at the origin.* The result is discussed in Vivanti's 1907 report on entire functions but without attributing it to any particular person. I find it established in a paper [133] of Cesaro (1884) not mentioned in Vivanti's bibliography. Cesaro's exceedingly simple proof is as follows: For every zero of the derivative of such a canonical $F(z)$ of genre p we have

$$0 = \frac{F'(z)}{F(z)} = \sum_i \left(\frac{1}{z - \alpha_i} + \frac{1}{\alpha_i} + \frac{z}{\alpha_i^2} + \cdots + \frac{z^{p-1}}{\alpha_i^p} \right),$$

which may be written

$$0 = z^p \sum_i \frac{1}{\alpha_i^p (z - \alpha_i)} . \tag{24}$$

Now when all the roots α_i of $F(z)$ are real and the genre p is an even integer, the coefficient of i in every term of Σ in (24) has a sign opposite to that of y. Therefore the equation can not be satisfied for an imaginary value of z. On the other hand, if p is an odd integer, we may put the equation into the form

$$0 = z^p \sum_i \frac{\alpha_i z}{\alpha_i^{p+1}(z - \alpha_i)}, \tag{25}$$

and then the same conclusion follows, inasmuch as for real α the imaginary part of the coefficient of $\alpha z/(z-\alpha)$ is

$$-\frac{i\alpha^2 y}{(x - \alpha)^2 + y^2},$$

which again is opposite in sign to y. In consequence, $F'(z)$ has no imaginary roots. Furthermore, the derivative of each term under the summation sign in (24) for even p and in (25) for odd p is negative for real $z \neq \alpha_i$. In each case Σ is therefore a monotone non-increasing function and consequently between any two roots α_i of $F(z)$ on the same side of the origin there lies one and only one of its derivative.

It must not be thought that the conclusions obtained concerning the reality of roots of the derivative will carry over to higher derivatives, for the suppositions underlying the proof regarding the given entire function ordinarily do not hold regarding the derivative. If, for instance, the given function is canonical, the derived function is not necessarily so. Again, a function of genre p has a derivative of genre p at most, and an example has been given of a function of genre p with a derivative of genre $p-1$. The question therefore arises: What are then the necessary and sufficient conditions to be imposed upon a given entire function possessing only real roots in order that its derivatives of all

orders shall likewise have only real roots? This very difficult question has recently been tackled by Pólya.

The nature of the answer seems to have been suggested by the following theorem.

THEOREM OF LAGUERRE.* *If a sequence of polynomials whose roots are all real converges over every finite region of the z-plane,† the limit of the sequence is an entire function of genre* 0 *or* 1, (*by the above theorem of Hurwitz, possessing only real zeros*), *or the same multiplied by a factor* e^{cz^2} *in which c is real. Furthermore, if the roots of the polynomials are all of like sign, the limit, except for a possible factor* e^{az} *with real a, will be a function of genre* 0 (*possessing only roots of the same sign*).

This theorem and others to be presently mentioned seem to me noteworthy in that they establish the genre of the function not on the basis of the asymptotic character of the coefficients or roots but by use of functional properties.

The derivatives of the polynomials in the theorem are of the same character as the polynomials themselves, having likewise only real roots (respectively real roots of the same sign), and also by a well known theorem on analytic functions converge uniformly over every finite region of the z-plane. The conclusions of the theorem can therefore be applied to succeeding derivatives of the entire function. Thus on the given hypothesis all derivatives have only real roots (respectively real roots of the same sign).

This application of Laguerre's Theorem stops short, however, of answering the question proposed. For, on the one hand, we are not sure that all real entire functions with real zeros are obtainable as limits of sequences of polynomials with real roots. On the other hand, there exist *unreal* entire functions possessing only real zeros. To get such a function we have merely to take in (20) an unreal $G(z)$ along with a canonical product Π having only real zeros. Further-

* *Oeuvres*, vol. 1, 1882, pp. 174–177.

† Apparently uniform convergence is intended.

more, the possibility that all the succeeding derivatives should have only real roots, if any, can be actually realized. The function $c'(e^{ibz}-c)$, in which we take b real and $|c|=1$, is an unreal entire function possessing an infinite periodic system of real zeros and no others, while its successive derivatives have no zeros at all. It is easy to see that it is of genre 1. Pólya cites $c'e^{icz}$ (c real) as a second instance, this being an unreal function which with its successive derivatives is devoid of imaginary zeros. He [120a] conjectures that with just these two exceptions all entire functions which together with their derivatives have only real zeros must be of the form

$$cz^r e^{-\gamma^2 z^2+dz}\prod(1-d_n z)e^{d_n z}, \tag{26}$$

in which all constants except c are real and Σd_n^2 is convergent. Note that we have here exactly the first kind of functions mentioned in the above Theorem of Laguerre with the added restriction that the coefficient of z^2 in the exterior exponential factor is negative or zero. This conjecture Pólya was able to prove on the supposition that the entire function is of finite genre and has only a finite number of zeros. It can then be written in the form $e^{Q(z)}P(z)$ where P and Q are polynomials, so that we have then a purely polynomial problem. He showed further that when all the roots of the entire function (except $z=0$) are of positive sign, we must take $\gamma=0$ and a non-positive d; accordingly the function except for such a possible exterior factor e^{dz} is of genre zero. In 1923 Alander [124] extended this conclusion to any entire function of finite genre without the limitation to a finite number of zeros. This he did by showing that the function of finite genre, unless it has one of the two exceptional forms above noted, can have only a finite number of zeros, so that Pólya's conclusion is applicable.

The sufficiency of the form (26) for the reality of the roots of all the successive derivatives Pólya established neatly by setting up a sequence of polynomials

$$\phi_n(z) = cz^r\left(1-\frac{\gamma^2z^2}{n}\right)^n \cdot\left(1+\frac{d+d_1+\cdots+d_n}{m_n}\right)^{m_n}\prod_{\nu=1}^{n}(1-d_\nu z),\ (n=1,2,\cdots),$$

which with proper choice of the positive integers m_n converge uniformly to the function (26) in every finite region of the z-plane. The character of the limiting function then follows by Laguerre's Theorem given above.

It is interesting to note another closely related result of Pólya [136]. When the entire function has the form $e^{Q(z)}P(z)$ in which $Q(z)$ and $P(z)$ are polynomials, the number of imaginary roots of its nth derivative increases indefinitely with n if $Q(z)$ is of higher degree than 2, and likewise if it is of degree 2 unless the coefficient of z^2 in $Q(z)$ is negative.

The interesting theorem of Laguerre above quoted was based upon the hypothesis of a sequence of polynomials with real roots which converged uniformly in every finite region of the plane. Pólya [120b] has notably lightened the hypothesis by showing that it is sufficient to demand that they converge uniformly in some finite region containing the origin. One thinks here of Stieltjes's Theorem that if a sequence of analytic functions converges uniformly in any given region and is bounded in absolute value over any larger region embracing the given region, then it also converges uniformly over this larger region. Pólya does not use this theorem but is able to base the extension of the region of uniform convergence directly upon the reality of the roots provided the limit is not identically zero. In a subsequent paper [130] he lightens even this latter requirement and asks only that the roots of the polynomials shall all lie within some sector having its vertex at the origin.

When the polynomials under consideration are segments $S_n(z)=a_0+a_1z+\cdots+a_nz^n$ of a power series, these results of Pólya follow immediately from a remarkable theorem of Jentzsch [141]. If, namely, the series has a finite radius of convergence $R>0$, every point of the circle of convergence

is a limit point of the set of roots of the equations $S_n(z)=0$, $(n=1, 2, \cdots)$. Suppose now that there is a sector with vertex at the origin free from the roots of $S_n(z)=0$. Then if the polynomial segments $S_n(z)$ converge in a region including the origin, it will follow that R is infinite, since otherwise the theorem is violated. Later Szegö [142] has given an extension of the theorem in which the sequence is no longer limited to a polynomial sequence obtained by segmenting a power series.

Jentzsch [143] has obtained a theorem which derives the existence and genre of an entire function from suppositions regarding the roots of its segments without making the slightest demand regarding the convergence of the sequence $S_n(z)$, $(n=1, 2, \cdots)$. Denote by $\beta_{1,n}, \beta_{2,n}, \cdots, \beta_{n,n}$ the n roots of $S_n(z)$. Then if from and after some fixed value of n these roots satisfy the inequality

$$\sum_{i=1}^{n} \frac{1}{|\beta_{in}|^{\rho}} < M, \qquad (n = r, r+1, \cdots), \tag{27}$$

where ρ is some positive integer and M a fixed positive number, the series will converge over the entire plane and its limit will be an entire function of genre equal to or less than ρ. As a particular application of this result consider any series whose segments $S_n(z)$ have only real roots. By the theory of symmetric functions we have

$$\sum_{i=1}^{n} \frac{1}{|\beta_{in}|^{2}} = \frac{a_1^2 - 2a_0a_2}{a_0^2}.$$

Here $\rho=2$, and M is any number greater than the right-hand member. It follows, therefore, if the polynomial sections $S_n(z)$ of a formally given series have only real roots, the series converges to a limit, and the limit is an entire function of genre 2 at most. Pólya [120b] (1913) cuts down the genre to 1 at most, and to 0 when the roots are all of the same sign. Jentzsch states also, without giving proof, in his 1914 Comptes Rendus note [144], that his conclusion will hold

also for any sequence of polynomials $a_{n0}+a_{n1}z+\cdots+a_{nn}z^n$, $(n=1, 2, \cdots)$, whose roots satisfy the inequality (27), provided that it converges uniformly in some region of the plane. In the same year Lindwart and Pólya [128] considered such a sequence under the same supposition but with the added limitation (presumably unnecessary) that the region of uniform convergence shall include the origin, and proved not only that it has a limit which is an entire function of genre equal to or less than ρ but that also the limit is actually a function of genre $\rho-1$ at highest except for a possible factor e^{cz^ρ}.

I turn lastly to a topic as yet but too little worked, the determination of the conditions which must be imposed upon the coefficients of an entire function to secure a specified distribution of its roots. In the case of a real polynomial of the nth degree, it has been long known that if its roots are *distinct*, the number of pairs of imaginary roots is equal to the number of changes of sign in the sequence

$$(28)\qquad s_0, \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}, \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}, \cdots, \begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \cdot & \cdot & \cdots & \cdot \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{vmatrix},$$

where $s_0=n$, and s_i denotes the sum of the ith power of the roots. The roots will be all real if and only if all of these determinants are positive. If the members of (28) are all positive when we advance every subscript by unity, we have, as Grommer [121] has shown, the necessary and sufficient condition that the roots should all be distinct and positive, provided that there is no root at the origin.

With this proviso, it is obviously permissible to replace the roots by their reciprocals, or, in other words, to substitute s_{-i} for s_i. This is necessary before we can think of generalizing the condition so as to get a corresponding result for entire functions. But the indefinitely continued sequence (28) is obviously unusable even for an entire function of

genre 0 with an infinite number of roots, for though the sum of the ith powers of the reciprocals of the roots will converge for all values of i, the element $s_0=n$ increases indefinitely with the degree of the polynomial. Fortunately, in place of (28) we can use the sequence

$$(29)\quad s_{-2},\ \begin{vmatrix} s_{-2} & s_{-3} \\ s_{-3} & s_{-4} \end{vmatrix},\ \cdots,\ \begin{vmatrix} s_{-2} & s_{-3} & \cdots & s_{-n-1} \\ \cdot & \cdot & \cdots & \cdot \\ s_{-n-1} & s_{-n} & \cdots & s_{-2n} \end{vmatrix}$$

to test the reality of the roots of a polynomial of nth degree. The roots are all real when and only when the terms of the sequence (29) are either all positive or are positive up to a certain member inclusive, after which they are all zero [121, p. 114]. The elements s_{-i} therein contained have a significance for entire functions of genre 0 or 1, the sum of the ith powers of the reciprocals of the roots being convergent for $i\geqq 2$. According to Grommer, Hurwitz considered the question whether the roots must all be real when the terms of (29) continued indefinitely are all positive for such a function. But the elements s_{-i} in (29) may be assigned a significance for any entire function. If, in fact, we take an entire function $F(z)=a_0+a_1z+\cdots$ without a zero at the origin and expand the negative of its logarithmic derivative into a series

$$(30)\qquad -\frac{F'(z)}{F(z)}=s_{-1}+s_{-2}z+s_{-3}z^2+\cdots,$$

the successive coefficients are connected by linear recurrence formulas in which both the s_{-i} and the a_i appear linearly. When the function is of genre 0 the s_{-i} are actually equal to the sum of the ith powers of the reciprocals of the roots by virtue of the relation

$$-\frac{F'(z)}{F(z)}\equiv -\sum_{i=1}^{\infty}\frac{1}{z-\alpha_i}=\sum_{i=1}^{\infty}\left(\frac{1}{\alpha_i}+\frac{z}{\alpha_i^2}+\cdots\right).$$

Whether or not this be the case, let the determinant sequence (29) continued indefinitely be formed with elements taken from the series (30). With this generalization of the s_{-i}

Grommer took up the problem at Toeplitz's suggestion and proved that when the indefinitely continued sequence (29) is merely restricted by the requirement that all its terms shall be positive, the entire function will be of genre 0 or 1 except possibly for a factor $e^{-\gamma^2 z^2}$ (γ real). His proof is complicated, involving the use of quadratic forms, continued fractions and Stieltjes integrals. Later, in 1920, Kritikos [122] under the instigation of Pólya re-derived the result by a simpler method. Grommer derives also a more general theorem, according to which the necessary and sufficient condition that a real entire function $F(z)$, for which $F(0) \neq 0$, shall have an infinite number of roots all real and simultaneously shall be an entire function of genre $2m-1$ or less, except for a factor $e^{-\gamma^2 z^{2m}}$ (γ real), is that all the determinants

$$\begin{vmatrix} s_{-2m} & s_{-2m-1} & \cdots & s_{-2m-k+1} \\ \cdot & \cdot & \cdots & \\ s_{-2m-k+1} & s_{-2m-k} & \cdots & s_{-2m-2k+2} \end{vmatrix}, \quad (k = 1, 2, \cdots),$$

shall be positive.

It may not be without interest to say a word or two regarding the opposite possibility that all the roots shall be imaginary. Long ago I showed [2] very easily that if

$$(31) \quad a_0, \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}, \cdots, \begin{vmatrix} a_0 & a_1 \cdots a_{n-1} \\ \cdot & \cdots \cdot \\ a_{n-1} a_n \cdots & a_{2n-2} \end{vmatrix},$$

$$(a_i = 0 \text{ for } i > n),$$

are all positive, all the roots of the polynomial

$$P_n(z) = a_0 + a_1 z + \cdots + a_n z_n$$

are imaginary if n is even, or all but one if n is odd. The condition is sufficient but not necessary. It follows that if for an entire function $a_0 + a_1 z + a_2 z^2 + \cdots$ all the terms of the sequence (31), indefinitely continued, are positive, it is an imaginary function with only imaginary zeros. A diver gent power series may also be consistent with the requirement

of the positive character of (31), as will be seen by increasing a_0, a_2, a_4, sufficiently rapidly.

This I trust is sufficient to indicate the kind of theorems which are being obtained by extension of the theory of equations into the province of the theory of analytic functions. The mine has been yet imperfectly worked and much ore remains to be extracted.

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PART I. ON THE ROOTS OF POLYNOMIALS

A. On the Number of Real or Imaginary Roots; Number in an Interval.

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(b) *Remarque sur les zéros des fonctions entières*, Bulletin de la Société Mathématique, vol. 32 (1904), pp. 65–67.

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136. G. Pólya, *Ueber die Nullstellen sukzessiver Derivierten*, Mathematische Zeitschrift, vol. 12 (1922), pp. 36–60.

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139. M. Biernacki, *Sur le déplacement des zéros des fonctions entières par leur dérivation*, Comptes Rendus, vol. 175 (1922), pp. 18–20.

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141. R. Jentzsch, (a) *Untersuchungen zur Theorie der Folgen analytischer Funktionen*, Dissertation, Berlin, 1914; and Acta Mathematica, vol. 41 (1918), pp. 219–251.

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(Jentzsch proves that every point on the circle of convergence of a Taylor's power series is a limit point of roots of its segments. In No. 140 Lukacs showed that there was at least one such limit point on the circle. An extension of Jentzsch's result is given by Szegö in the following article.)

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143. R. Jentzsch, *Sur l'extension d'un théorème de Laguerre*, Comptes Rendus, vol. 158 (1914), pp. 780–782.

144. G. Pólya, *Ueber das Graeffesche Verfahren*, Zeitschrift für Mathematik und Physik, vol. 63 (1915), pp. 275–290.

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The University of Wisconsin

AN ANNOUNCEMENT REGARDING FACTOR STENCILS

BY D. N. LEHMER

An edition of fifty sets of Factor Stencils has been completed and will shortly be distributed by the Carnegie Institution of Washington under whose auspices the work of construction has been done. The device is intended to facilitate the finding of factors of numbers of an order as high as two billion and a half.

The theory of the stencils is based on the well known fact that if R is known to be a quadratic residue of a number N then the factors of N are to be found in certain linear forms. Thus if -1 is known to be a residue of N, the factors of N are all of the form $4n+1$; and if 2 is a residue of N, the factors of N are of the form $8n \pm 1$; and so on. These linear forms may be combined and a set of linear forms deduced from them to limit the number of trials necessary to determine the character of the number N.

As the number R increases, however, the number of forms to be considered also increases so that the combination of the forms corresponding to two numbers R such as 113 and 199 would involve some 11,088 resulting forms. The straightforward combination of the linear forms becomes impossible except for very small values of R. Nevertheless large values of R will exclude the same proportion of trials as small values, and the stencil device is intended to make it possible to use values of R as high as ± 238. As each residue discovered serves to reject approximately half the number of trials, it is seen that for a number of the order of 2,000,000,000 where some 5,000 trial divisors must be examined, one residue reduces the number to about 2,500 trials; two to about 1,250; three to 625; four to 312; five to 78; six to 39; seven to 20; eight to 10; nine to 5; and ten to 2. The finding, therefore of

ten quadratic residues of N ought to be sufficient to find the factors of a number of the order of two billion. For smaller numbers, of course, a smaller list will serve. The only difficulty lies in the combination of the linear forms belonging to the residues. This difficulty is completely solved by the factor stencils. One does not seek to find the resultant forms of all the forms belonging to the residues, but only the primes that may lie in them. This is accomplished by a plan which is believed to be entirely new, and the work of examining a number is practically complete as soon as the list of residues is found. The construction of the Factor Stencils is as follows.

The first page of the list of primes published by the Carnegie Institution of Washington in 1914 contains 5,000 primes beginning with 1 and ending with 48,593. Any composite number not greater than 2,361,279,649, which is the square of 48,593, will have at least one of its factors in this list of primes. The list is arranged in fifty columns with one hundred rows in each column. A sheet of paper for a stencil is then ruled to show 5,000 cells, 50 columns and 100 rows. For a given R, holes are punched in those cells corresponding to primes which have R for a quadratic residue. Thus for $R=-1$ holes appear in the cells corresponding to the position in the list of primes of the numbers 1, 5, 13, 17, 29, 37, $\cdots$, and for $R=2$ the holes appear in the cells corresponding to the numbers 1, 7, 17, 23, 31, 41, 47, 71, 73, $\cdots$; the first set all having -1 for a residue and the second all having 2 for a residue. If these two stencils are superposed it is clear that those holes which appear through both indicate primes which are in both sets and have both -1 and 2 for residues. The first holes that thus appear are those corresponding to the primes 17 and 31. If now the stencil for a third residue is laid on the other two approximately half the remaining holes are covered and the holes that shine through the three stencils indicate primes that have all three numbers for quadratic residues. The problem of finding the factors of a number within the range of the first 2,000,000,000 numbers is reduced to the finding of some eight or ten quadratic residues.

In the book which accompanies the stencils various methods of finding quadratic residues are discussed and the method by means of the expansion of the square root of the number in a continued fraction is found to be by far the most effective. Various examples are given illustrating the power of the stencils, and a reproduction of the first page of the list of primes accompanies the work.

The plan of the stencils was first conceived in November 1924 and has been carried on since then under grants from the Carnegie Institution of Washington. Plans for the distribution of the sets are not yet completed, but every effort will be made to place them where they will be of most use.

THE UNIVERSITY OF CALIFORNIA

ON A PROBLEM IN THE THEORY OF GROUPS ARISING IN THE FOUNDATIONS OF INFINITESIMAL GEOMETRY*

BY H. P. ROBERTSON AND H. WEYL

In another paper in this issue,† the fundamental problem of infinitesimal geometry is formulated as the problem of uniquely associating with an arbitrary coordinate system on the manifold M a normal coordinate system on the tangent plane T_P by means of the fundamental coefficients of displacement on M.

The importance of the other aspect of this problem raised by O. Veblen and H. P. Robertson, yet remains: to associate a transformation of the given group $\mathfrak{G}$ with an arbitrary tranformation of the coordinates x in such a way that it gives rise to a representation by $\mathfrak{G}$, that is, that to composition of arbitrary transformations of x corresponds composition of the associated transformations of $\mathfrak{G}$. From

* Presented to the Society, June 21, 1929.

† This issue, pp. 716–725.

this standpoint the problem is a pure group-theoretic one which is not at all concerned with a "connection" on M. We consider in the following the projective group as an example.

More precisely, the problem is the following. Take the point P under consideration as the origin of coordinates, that is, consider the relative coordinates $x_k - x_0^i$, where $P = P(x_0)$. An arbitrary transformation of these relative coordinates x^i has the form

$$(1) \qquad s: \quad \bar{x}^i = a_k^i x^k + \tfrac{1}{2} a_{kl}^i x^k x^l + \cdots .$$

The associated n-dimensional projective transformation σ, which shall also leave the origin fixed, is expressed in homogeneous coordinates η in accordance with the normalization (4) of the paper mentioned above. We take as coefficients a_k^i of σ the coefficients a_k^i of (1); the coefficients a_k are then to be determined. The condition imposed is that the association $s \to \sigma$ shall yield a representation. We may add the additional condition that σ depend only on the coefficients a_k^i and a_{kl}^i of s of first and second orders.

We are then naturally led to the problem of representing the group $\mathfrak{G}$ of all transformations by linear transformations, whereby the accessory condition that the transformation corresponding to s shall depend on the terms of the first two or first three orders etc. may be imposed. This problem is to be attacked as follows. We first restrict ourselves to the subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$ consisting of linear transformations

$$\bar{x}^i = a_k^i x^k .$$

We know the representations of $\mathfrak{G}_0$ and we shall assume that the given representation $s \to \sigma$ of $\mathfrak{G}$ by N-dimensional matrices σ decomposes into known irreducible representations when considered as a representation of $\mathfrak{G}_0$ (that is certainly the case when we restrict ourselves to the unimodular transformations of $\mathfrak{G}_0$; we take it to be the case for the full linear group for the sake of simplicity). Let these irreducible constituents be of orders o, o', $\cdots$, respectively; i.e. to the infinitesimal dilatation $dx^i = x^i$ correspond the dilatations

$d\xi^\iota = o\xi^\iota$, $d\xi^\iota = o'\xi^\iota$, $\cdots$, respectively in these representations. (It follows from the general theory that in an irreducible representation a dilatation must correspond to a dilatation in the original group.)

Furthermore, we employ Lie's infinitesimal method. To an infinitesimal transformation s

$$dx^i = a_k^i x^k + \frac{1}{2!} a_{kl}^i x^k x^l \tag{2}$$

corresponds an infinitesimal transformation

$$d\xi^\iota = \alpha_\kappa^\iota \xi^\kappa \tag{3}$$

of the given representation. But the α_κ^ι are now *linear* homogeneous functions of the a_k^i, a_{kl}^i, $\cdots$. The condition that σ be a representation requires in addition that to the commutator of two transformations s corresponds the commutator of the corresponding transformations ξ. We consider in particular the two transformations

$$s: \quad dx^i = \frac{1}{2!} a_{kl}^i x^k x^l + \cdots \quad \text{and} \quad C: \quad d'x^i = c_k^i x^k. \tag{4}$$

If the matrices of the corresponding σ (3) be A and Γ, the infinitesimal σ with matrix $A\Gamma - \Gamma A$ shall correspond to the commutator of (4). We proceed in three steps, by first taking as C the dilatation

$$d'x^i = x^i, \quad \text{that is,} \quad c_k^i = \delta_k^i,$$

then the more general linear "principal transformation" $d'x^i = \lambda_i x^i$ (not summed!) with arbitrary λ_i, and finally the unrestricted C.

The commutator of (2) and the dilatation is obtained by multiplying the term of nth degree in (2) by $(n-1)$. The set of variables ξ^ι is divided into subsets the members of which are transformed among themselves under Γ, and in particular are only multiplied by o, o' etc. by the transformation $\Gamma = \Gamma_d$ which corresponds to the dilatation. The rectangle $[\Lambda]$ of the matrix A, in which the set o intersects the

set o', is multiplied by $o'-o$ on going over to $A\Gamma_d-\Gamma_d A$ from A. The $\alpha_\kappa{}^i$ in this rectangle must consequently be multiplied by $o'-o$ when the $a_k{}^i$, a_{kl}^i, etc. are replaced by $0\cdot a_k{}^i$, $1\cdot a_{kl}^i$ etc. Consequently if $o'-o$ is not a whole number, 0, 1, 2, $\cdots$, [A] must vanish identically, and if $o'-o=n-1$, [A] can depend only on the terms of order n in (2). In particular the square [A] of A in which the set o intersects itself can only depend on the $a_k{}^i$.

We can choose the coordinates in representation space in such a way that to the principal transformations $dx^i=\lambda_i x^i$ correspond principal transformations $\xi^i\to\Lambda_i\xi^i$; the "weights" Λ_i are linear forms in the λ_i. In the commutator of (2) with this principal transformation a_{kl}^i is replaced by

$$a_{kl}^i\ldots(\lambda_k+\lambda_l+\cdots-\lambda_i)$$

while the corresponding process for the σ changes $\alpha_\kappa{}^i$ into $\alpha_\kappa{}^i(\Lambda_\kappa-\Lambda_i)$. Consequently the $\alpha_\kappa{}^i$ can depend only on those $a_{kl}^i\cdots$ for which $\lambda_k+\lambda_l+\cdots-\lambda_i$ is identical with $\Lambda_\kappa-\Lambda_i$.

Equivalent representations, which differ only in the choice of the coordinate system in which they are expressed, shall naturally be considered as the same. But as soon as the irreducible representations into which the given representation of $\mathfrak{G}_0$ is decomposed is fixed the coordinate system is fixed except for multiplication of the variables of each subset by a non-vanishing arbitrary constant.

Let us return to the case considered above, in which $N=n+1$ and the $(n+1)$-dimensional representation of $\mathfrak{G}_0$ is decomposed into the n-dimensional $s\to s$ and the one-dimensional which associates the identity $\xi^0=\xi^0$ with every s. Here $o=1$, $o'=0$; we must consequently have in (3)

$$\alpha_k{}^i=a_k{}^i,\qquad \alpha_0{}^0=1,\qquad \alpha_0{}^i=0,$$

and $\alpha_k{}^0$ depends only on the terms a_{rs}^i of second order. Furthermore, $\alpha_k{}^0$ must be linear form in those a_{rs}^i only for which $\lambda_r+\lambda_s-\lambda_i=\lambda_k$. We must therefore have

$$\alpha_k^0=\sum_i H_{ik}a_{ik}^i$$

where the H_{ik} are constants independent of s. In order to determine the H_{ik} we now consider the commutator of

$$dx^i = \tfrac{1}{2}a^i_{kl}x^k x^l \quad \text{and} \quad d'x^i = c_k{}^i x^k$$

and obtain

$$\sum_{i,r} H_{ik}(a^i_{rk}c_i{}^r + a^i_{ir}c_k{}^r - a^r_{ik}c_r{}^i) = \sum_{i,r} c^r_k H_{ir} a^i_{ir}$$

or

$$\sum_{i,r} a^i_{rk}(H_{ik} - H_{rk})c_i{}^r = \sum_{i,r} b^r_k(H_{ir} - H_{ik})a^i_{ir}.$$

Equating the coefficients of $c_i{}^k$ on both sides for $i \neq k$ we see that $H_{ik} = H_{kk}$; H_{ik} does not depend on i and can be written H_k. The equation then reduces to

$$\sum (H_r - H_k)c^r_k = 0$$

which tells us that H_k is independent of k, $H_k = H$. Hence

$$\alpha^0_k = H \sum_i a^i_{ik}.$$

When $H \neq 0$ we can multiply the variable ξ^0 by an appropriate constant in such a way that in this new coordinate system

$$H = -\frac{1}{n+1}.$$

We thus find that there are two distinct possibilities: (1) $\alpha^0_k = 0$ or the affine tangent plane, (2) the one obtained by Robertson and, from another standpoint, in the paper by H. Weyl on pages 716–725 of this issue, that is, the semi-osculating projective plane.

PRINCETON UNIVERSITY

NOTE ON LINEAR TRANSFORMATIONS OF n-ICS IN m VARIABLES*

BY A. D. CAMPBELL

Let us consider the n-ic in m variables

$$F(x_1, x_2, \cdots, x_m) = 0. \tag{1}$$

If we subject (1) to the linear transformation

$$\begin{aligned} \rho x_1 &= a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1m}x_m', \\ \rho x_2 &= a_{21}x_1' + a_{22}x_2' + \cdots + a_{2m}x_m', \cdots, \\ \rho x_m &= a_{m1}x_1' + a_{m2}x_2' + a_{m3}x_3' + \cdots + a_{mm}x_m', \end{aligned} \tag{2}$$

we obtain

$$\begin{aligned} &F(a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1m}x_m', a_{21}x_1' + a_{22}x_2' \\ &\quad + a_{23}x_3' + \cdots + a_{2m}x_m', \cdots, a_{m1}x_1' + a_{m2}x_2' \\ &\quad + a_{m3}x_3' + \cdots + a_{mm}x_m') = 0. \end{aligned} \tag{3}$$

Note that in the expansion of (3) the coefficient of the term in $x_i'^n$, $(i=1, 2, 3, \cdots, m)$, is $F(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$. A necessary and sufficient condition for this coefficient to vanish is that the point $P_i(a_{1i}, a_{2i}, \cdots, a_{mi})$ shall lie on the geometric locus of (1). To obtain the coefficient of such a term as $x_i'^r x_j'^{n-r}$ in the expansion of (3) we can put

$$\begin{aligned} &x_i' x_j' \neq 0, \; x_1' = x_2' = x_3' = \cdots = x_{i-1}' \\ &\quad = x_{i+1}' = x_{i+2}' = \cdots = x_{j-1}' = x_{j+1}' = \cdots = x_m' = 0, \end{aligned}$$

then use Taylor's Expansion on

$$\begin{aligned} &F(a_{1i}x_i' + a_{1j}x_j', a_{2i}x_i' + a_{2j}x_j', a_{3i}x_i' \\ &\quad + a_{3j}x_j', \cdots, a_{mi}x_i' + a_{mj}x_j') \equiv F(X_1 + X_1', \\ &X_2 + X_2', X_3 + X_3', \cdots, X_m + X_m'), \end{aligned} \tag{4}$$

* Presented to the Society, August 29, 1929.

where $X_1=a_{1i}x_i'$, $X_1'=a_{1j}x_j'$, $X_2=a_{2i}x_i'$, $X_2'=a_{2j}x_j'$, etc. We find the coefficient of $x_i'^r x_j'^{n-r}$ in the group of terms*

$$\frac{1}{(n-r)!}\left(\frac{\partial F}{\partial X_1}X_1' + \frac{\partial F}{\partial X_2}X_2' + \frac{\partial F}{\partial X_3}X_3' + \cdots + \frac{\partial F}{\partial X_m}X_m'\right)^{(n-r)}$$

$$\equiv \frac{(x_i'^r x_j'^{n-r})}{(n-r)!}\left(\frac{\partial F}{\partial a_{1i}}a_{1j} + \frac{\partial F}{\partial a_{2i}}a_{2j} + \frac{\partial F}{\partial a_{3i}}a_{3j} + \cdots + \frac{\partial F}{\partial a_{mi}}a_{mj}\right)^{(n-r)}, \tag{5}$$

where $\partial F/\partial a_{1i}$ means $\partial F/\partial X_1$ with X_1 replaced by a_{1i}, X_2 by a_{2i}, X_3 by a_{3i}, $\cdots$, X_m by a_{mi}, and similarly for $\partial F/\partial a_{2i}$, $\partial F/\partial a_{3i}$, $\cdots$, $\partial F/\partial a_{mi}$. Hence we may conclude that a necessary and sufficient condition for the vanishing of the coefficient of $x_i'^r x_j'^{n-r}$ in the expansion of (3) is that the point $P_j(a_{1j}, a_{2j}, a_{3j}, \cdots, a_{mj})$ shall lie on the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, \cdots, a_{mi})$ with respect to the locus of (1).

We obtain the coefficient of the term in $x_i'^r x_j'^s x_k'^t \cdots x_l'^u x_p'^v$ (where $r+s+t+\cdots+u+v=n$) in the expansion of (3) by the following device. We can write (3) as

$$F(Y_1+Y_1', Y_2+Y_2', Y_3+Y_3', \cdots, Y_m+Y_m') = 0, \tag{6}$$

where

$$Y_1 = a_{1i}x_i', Y_1' = a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1i-1}x_{i-1}' + a_{1i+1}x_{i+1}' + a_{1i+2}x_{i+2}' + \cdots + a_{1m}x_m',$$

$$Y_2 = a_{2i}x_i', Y_2' = a_{21}x_1' + a_{22}x_2' + \cdots + a_{2i-1}x_{i-1}' + a_{2i+1}x_{i+1}' + \cdots + a_{2m}x_m',$$

etc. We take the collection of terms

$$\frac{1}{(n-r)!}\left(\frac{\partial F}{\partial Y_1}Y_1' + \frac{\partial F}{\partial Y_2}Y_2' + \frac{\partial F}{\partial Y_3}Y_3' + \cdots + \frac{\partial F}{\partial Y_m}Y_m'\right)^{(n-r)} \tag{7}$$

* See Goursat-Hedrick *Mathematical Analysis*, vol. 1, pp. 107–108. For the Galois fields, see A. D. Campbell, *The polar curves of plane algebraic curves in the Galois fields*, this Bulletin, vol. 34 (1928), pp. 361–363. The methods of this paper may be readily generalized to the polars of an n-ic in m variables.

in the expansion of (6). All the terms with $x_i'^r$ as a factor must come from (7). We can write (7) in the form

$$\frac{x_i'^r}{(n-r)!}\left(\frac{\partial F}{\partial a_{1i}}Y_1' + \frac{\partial F}{\partial a_{2i}}Y_2' + \frac{\partial F}{\partial a_{3i}}Y_3' + \cdots + \frac{\partial F}{\partial a_{mi}}Y_m'\right)^{(n-r)}. \tag{8}$$

If we equate (8) to zero we obtain the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$ with respect to the locus of (1). We can also write (7) in the form

$$\frac{x_i'^r}{(n-r)!}F'(Y_1', Y_2', Y_3', \cdots, Y_m'), \tag{9}$$

where F' is a function of the $(n-r)$th degree. We put

$$Y_1' = Z_1 + Z_1', Y_2' = Z_2 + Z_2', \cdots, Y_m' = Z_m + Z_m', \tag{10}$$

where

$$\begin{aligned} Z_1 = a_{1j}x_j', Z_1' = {} & a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \cdots + a_{1i-1}x_{i-1}' \\ & + a_{1i+1}x_{i+1}' + a_{1i+2}x_{i+2}' + \cdots + a_{1j-1}x_{j-1}' + a_{1j+1}x_{j+1}' \\ & + a_{1j+2}x_{j+2}' + \cdots + a_{1m}x_m', Z_2 = a_{2j}x_j', \end{aligned}$$

etc. Expanding (9), we find that all the terms in the expansion of (3) that have the factors $x_i'^r$ and $x_j'^s$ must be in the collection of terms

$$\begin{aligned} & \frac{x_i'^r}{(n-r)!(n-r-s)!}\left(\frac{\partial F'}{\partial Z_1}Z_1' + \frac{\partial F'}{\partial Z_2}Z_2' + \frac{\partial F'}{\partial Z_3}Z_3' + \cdots + \frac{\partial F'}{\partial Z_m}Z_m'\right)^{(n-r-s)} \equiv \frac{(x_i'^r x_j'^s)}{(n-r)!(n-r-s)!}\left(\frac{\partial F'}{\partial a_{1j}}Z_1'\right. \\ & \left. + \frac{\partial F'}{\partial a_{2i}}Z_2' + \frac{\partial F'}{\partial a_{3j}}Z_3' + \cdots + \frac{\partial F'}{\partial a_{mj}}Z_m'\right)^{(n-r-s)}. \end{aligned} \tag{11}$$

If we equate (11) to zero we shall have the $(n-r-s)$th polar of the point $P_j(a_{1j}, a_{2j}, a_{3j}, \cdots, a_{mj})$ with respect to the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$ with respect to the locus of (1).

Next we take

$$Z_1' = W_1 + W_1', \; Z_2' = W_2 + W_2',$$
$$Z_3' = W_3 + W_3', \cdots, Z_m' = W_m + W'_m$$

in (11), where

$$W_1 = a_{1k}x_k', \; W_1' = a_{11}x_1' + a_{12}x_2' + \cdots + a_{1i-1}x_{i-1}'$$
$$+ a_{1i+1}x_{i+1}' + \cdots + a_{1j-1}x_{j-1}' + a_{1j+1}x_{j+1}' + \cdots$$
$$+ a_{1k-1}x_{k-1}' + a_{1k+1}x_{k+1}' + \cdots + a_{1m}x_m', \; W_2 = a_{2k}x_k',$$

etc. We repeat the above processes until we finally reach the collection of terms having all the factors $x_i'^r$, $x_j'^s$, $x_k'^t$, $\cdots$, $x_l'^u$, and $x_p'^v$. Therefore, we see that for the coefficient of the term $x_i'^r x_j'^s x_k'^t \cdots x_l'^u x_p'^v$ in the expansion of (3) to vanish we must have the point $P_p(a_{1p}, a_{2p}, a_{3p}, \cdots, a_{mp})$ on the $(n-r-s-t- \cdots -u-v)$th polar of $P_l(a_{1l}, a_{2l}, \cdots, a_{ml})$ with respect to the $\cdots$th polar of $\cdots, \cdots, \cdots,$ with respect to the $(n-r-s-t)$th polar of $P_k(a_{1k}, a_{2k}, \cdots, a_{mk})$ with respect to the $(n-r-s)$th polar of $P_j(a_{1j}, a_{2j}, \cdots, a_{mj})$ with respect to the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, \cdots, a_{mi})$ with respect to the locus of (1).

It is noteworthy that this discussion applies to the ordinary complex or real domains and also to the Galois fields.

SYRACUSE UNIVERSITY

NON-EXISTENCE THEOREMS ON THE NUMBER OF REPRESENTATIONS OF ARBITRARY ODD INTEGERS AS SUMS OF $4r$ SQUARES*

BY E. T. BELL

1. *Introduction.* The theorems stated in §3 and proved in §4 will be more significant if we first outline some known results and devise a definition which they suggest. In §5 an interesting problem is proposed, to which the method of this paper is at least partly applicable.

2. *Simplicity of the Number of Representations of an Integer as a Sum of $4r$ Squares.* Let n, r be given integers. The number $N(n, r)$ of one-rowed matrices $(x_1, x_2, \cdots, x_r)$ of integers $x_j \lesseqgtr 0$ $(j=1, 2, \cdots, r)$ such that $x_1^2+x_2^2+\cdots+x_r^2=n$ is called, as customary, the number of representations of n as a sum of r squares; $N(0, r)=1$. Henceforth let n be an arbitrary integer >0, and m an odd integer >0. Denote by $\zeta_j(n)$, $j \geqq 0$, the sum of the jth powers of all the divisors of n, $\zeta_j(0)=1$ by convention; and by $\xi_j(n)$ the sum of the jth powers of all the divisors $\equiv 1 \bmod 4$ of n minus the like sum for the divisors $\equiv 3 \bmod 4$. Write $(-1 \mid m) \equiv (-1)^{(m-1)/2}$. Then, either from the analysis of Bulyguin† or otherwise, it is known that the general structure of $N(m, 2r)$ is as follows:

$$N(m, 4r) = a\zeta_{2r-1}(m) + F_r(m),$$

$$N(m, 4r-2) = [b + c(-1 \mid m)]\xi_{2r-2}(m) + G_r(m),$$

where a, b, c are numerical constants (independent of m) different from zero; $F_j(m)=G_j(m)=0$ $(j=1, 2)$, and $F_r(m)$, $G_r(m)$, when $r>2$, are sums of homogeneous polynomials in the integers $y_1, y_2, \cdots, y_{2t} \gtrless 0$ such that

* Presented to the Society, June 20, 1929.

† Bulletin de l'Académie de St. Petersburg, 1914, pp. 389–404.

$$y_1^2+y_2^2+\cdots+y_{2t}^2=m,$$

where $t<2r$ for $F_r(m)$ and $t<2r-1$ for $G_r(m)$. For example,

$$F_8(m) = p\sum_8(y_1^8 - 28y_1^6y_2^2 + 35y_1^4y_2^4) + q\sum_{16}(y_1^4 - 3y_1^2y_2^2),$$

where $\sum_8, \sum_{16}$ refer respectively to all sets of solutions of

$$y_1^2 + y_2^2 + \cdots + y_8^2 = m, \qquad y_1^2 + y_2^2 + \cdots + y_{16}^2 = m,$$

respectively, and p, q are numerical constants different from zero. Precisely similar theorems hold for $N(2^\alpha m, 2r)$, $\alpha>0$; the constants a, b are then to be replaced by $a+A2^{(2r-1)\alpha}$, $b+B2^{2(r-1)\alpha}$ respectively, where A, B are numerical constants different from zero; $F_r(2^\alpha m)$, $G_r(2^\alpha m)$ replace $F_r(m)$, $G_r(m)$ and are defined in the same way as the latter with 2^α m in place of m throughout.

We shall call $a\zeta_{2r-1}(m)$, $[b+c(-1|m)]\,\xi_{2r-2}(m)$ the *simple parts* of $N(m, 4r)$, $N(m, 4r-2)$ respectively, and similarly for $N(2^\alpha m, 2r)$. The remaining parts will be called *compound*.

DEFINITION. *If $\alpha>0$, $\beta\geqq 0$ are constant integers, and if the compound part of $N(\alpha t+\beta, 2r)$ either vanishes or is identically zero for all integers $t\geqq 0$, $N(\alpha t+\beta, 2r)$ is said to be simple.*

For example, $N(t, 2r)$ is simple when and only when $r=1, 2, 3, 4$. In these cases the compound part vanishes identically; that is, it is absent. On the other hand, $N(2t, 12)$, $N(4t+3, 10)$ are simple because the compound parts of $N(n, 12)$, $N(n, 10)$, neither of which is identically zero, vanish when $n=2t, n=4t+3$ respectively. These six examples exhaust the known instances of simple $N(\alpha t+\beta, 2r)$.

We shall discuss the simplicity of $N(4t+\beta, 2r)$. Evidently the following enumeration of cases is exhaustive; it is made to fit the subsequent analysis:

$$N(4t+\beta,\ 4r-j), \qquad (\beta=0,1,2,3;\ \ j=0,2).$$

In other papers, cited presently, all cases except $N(4t+1, 4r)$, $N(4t+3, 4r)$ have been disposed of. We may therefore restrict the discussion to these.

3. *Theorems on* $N(m, 4r)$. We shall prove the following.

THEOREM 1. *If* $N(m, 4r)$, $m \equiv 1 \bmod 4$, *is simple,* $N(m, 4r) = 8r\zeta_{2r-1}(m)$.

THEOREM 2. *If* $N(m, 4r)$, $m \equiv 3 \bmod 4$, *is simple,*

$$N(m,4) = \frac{32r(2r-1)(4r-1)}{3(3^{2r-1}+1)}\zeta_{2r-1}(m).$$

THEOREM 3. *If* $m \equiv 1 \bmod 4$, *and* $N(m, 4r+4)$, $r>1$, *is simple, it is necessary (but not sufficient) that*

$$3 \cdot 5^{2r+2} = 512r^4 + 768r^3 + 352r^2 + 168r + 225.$$

THEOREM 4. *If* $m \equiv 3 \bmod 4$, *and* $N(m, 4r+4)$, $r>2$, *is simple, it is necessary (but not sufficient) that*

$$105(7^{2r+1}+1) = (512r^4 - 256r^3 - 32r^2 + 856r + 210)(3^{2r+1}+1).$$

THEOREM 5. $N(m, 4r+4)$ *is simple for no* $r>1$.

THEOREM 6. $N(m, 4r)$ *is simple when and only when* $r = 1, 2$.

To state the next theorem, let k be the modulus of the elliptic function sn u. Then, as is well known or easily seen by Maclaurin's theorem, the coefficient of $(-1)^r u^{2r+1}/(2r+1)!$ $(r \geqq 0)$ in the expansion of sn u is

$$S_{2r+1}(k^2) \equiv \sum_{t=0}^{r} s_t(r) k^{2t},$$

where the $s_t(r)$ $(t=0, 1, \cdots, r)$ are integers >0. The expansion being unique, the integers $s_t(r)$ are uniquely defined. Let $\binom{p}{j}$ denote the coefficient of x^j in the expansion of $(1+x)^p$ if $p>0$ and $0 \leqq j \leqq p$, and zero otherwise. Then

$$\sigma_\mu(r) \equiv \sum_{t=0}^{\mu} 2^{2t} \binom{2r+1-2t}{\mu-t} s_t(r)$$

is an integer >0 whenever $r \geqq 0$, $\mu \geqq 0$ are integers.

THEOREM 7. *If* $m \equiv 1 \bmod 4$, *a necessary and sufficient condition that* $N(m, 4r+4)$, $r \geqq 0$, *shall be simple, is that a constant* λ *(independent of* m*) shall exist such that*

$$\sigma_{2\mu}(r) = \lambda\binom{4r+4}{4\mu+1}, \qquad (\mu = 0,1,\cdots,r);$$

if such a λ *exists,* $\lambda N(m, 4r) = 2\zeta_{2r+1}(m)$.

THEOREM 8. *If* $m \equiv 3 \bmod 4$, *a necessary and sufficient condition that* $N(m, 4r+4)$, $r \geqq 0$, *shall be simple, is that a constant* λ (*independent of* m) *shall exist such that*

$$\sigma_{2\mu+1}(r) = \lambda\binom{4r+4}{4\mu+3}, \qquad (\mu = 0,1,\cdots,r);$$

if such a λ *exists,* $N(m, 4r+4) = 2\zeta_{2r+1}(m)$.

4. *Proofs.* It is readily seen that Theorems 7, 8 imply all the rest. We shall require the following explicit values of $s_t(r)$ $(t = 0, 1, 2, 3)$ for all integers $r \geqq 0$, due to Hermite:*

$$s_0(r) = 1\,;\ 2^4 s\,(r) = 3^{2r+1} - 8r - 3\,;$$

$$2^8 s_2(r) = 5^{2r+1} - (8r-4)3^{2r+1} + 32r^2 - 32r - 17\,;$$

$$2^{12} s_3(r) = 7^{2r+1} - (8r-12)5^{2r+1} + (32r^2 - 88r + 30)3^{2r+1} - \tfrac{1}{3}(256r^3 - 1056r^2 + 752r + 471).$$

Assume for a moment that Theorems 7, 8 are true. Then the value of λ is found by taking $\mu = 0$. Theorems 1, 2 result. Take $\mu = 0, 1$ in Theorems 7, 8; Theorems 3, 4 then follow by a simple but rather tedious calculation in an obvious manner. Theorem 5 follows from Theorems 3, 4 on remarking that for the appropriate (small) value of $r > 2$ the left-hand members of the equalities stated in Theorems 3, 4 become and remain greater than the respective right-hand members; the few values of r not thus rejected are thrown out by Theorems 7, 8. (The details of a similar calculation are given in the paper cited presently.) Theorem 6 is then immediate. It remains then only to prove Theorems 7, 8.

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* Oeuvres, vol. 3, p. 237; stated without indication of proof. As certain others of Hermite's values contain misprints, the above were calculated by the method of Gruder, Wiener Sitzungsberichte, vol. 126, IIa (1917), and found correct.

It was proved in a former paper* that

$$4\sum_m q^{m/2}\zeta_{2r+1}(m) = \sum_{t=0}^{r} s_t(r)\theta_2^{4t+2}\theta_3^{4r+2-4t}, \quad |q| < 1,$$

for all integers $r \geqq 0$, where $\sum_m$ refers to all odd integers $m > 0$, and θ_2, θ_3 are the usual elliptic theta constants. It was also proved that for each value of the integer $r \geqq 0$, the set of integer coefficients $s_t(r)$ $(t = 0, 1, \cdots, r)$ can not be replaced by any other set of $r+1$ integers > 0. Apply to the above the identities

$$\theta_2^2(q^{1/2}) = 2\theta_2\theta_3, \qquad \theta_3^2(q^{1/2}) = \theta_2^2 + \theta_3^2$$

from the transformation of the second order. A short reduction gives

$$2\sum_m q^{m/4}\zeta_{2r+1}(m) = \sum_{\mu=0}^{2r+1}\sigma_\mu(r)\theta_2^{2\mu+1}\theta_3^{4r+3-2\mu}.$$

From the uniqueness of the set of coefficients $s_t(r)(t = 0, 1, \cdots, r)$ and the manner in which the last identity has been obtained, it follows by a simple contradiction that the $\sigma_\mu(r)$ $(\mu = 0, 1, \cdots, 2r+1)$ can not be replaced by any other set of integers > 0. Replace q by q^4, and denote by $N(n, r, h)$ the number of representations of n as a sum of r squares, precisely h of which are odd. Since a sum of r squares, precisely h of which are odd, is $\equiv h \bmod 4$, we compare coefficients of like powers of q and separate cases modulo 4. Thus

$$m \equiv 1 \bmod 4\colon\ 2\zeta_{2r+1}(m) = \sum_{\mu=0}^{r}\sigma_{2\mu}(r)\frac{N(m, 4r+4, 4\mu+1)}{\binom{4r+4}{4\mu+3}};$$

$$m \equiv 3 \bmod 4\colon\ 2\zeta_{2r+1}(m) = \sum_{\mu=0}^{r}\sigma_{2\mu+1}(r)\frac{N(m, 4r+4, 4\mu+3)}{\binom{4r+4}{4\mu+3}}.$$

* To appear shortly in the Journal of the London Mathematical Society. The paper dealing with the remaining cases of $N(4t+\beta, 2r)$ has not yet been published.

By what precedes, the $\sigma_j(r)$ can not be replaced by any other sets of integers >0. But, obviously,

$$m \equiv 1 \bmod 4\colon\ N(m, 4r+4) = \sum_{\mu=0}^{r} N(m, 4r+4, 4\mu+1)\,;$$

$$m \equiv 3 \bmod 4\colon\ N(m, 4r+4) = \sum_{\mu=0}^{r} N(m, 4r+4, 4\mu+3).$$

Hence Theorems 7, 8 follow.

5. *Statement of a General Problem.* The theorem that $N(t, 2r)$ is simple when $r=1, 2, 3, 4$ is due to Eisenstein.* That $N(t, 2r)$ is simple only for those values of r does not seem to follow from his statements; the simplicity when and only when $r=1, 2, 3, 4$ can be proved readily by the methods of this paper. It follows from the theorems of the present paper and the others cited in §4 that $N(t, 2r)$, $N(4t+\beta, 2r)$ are simple only for the values of β, r stated in §1. Further, it is clear that the complete theory of the simplicity of $N(\alpha t+\beta, 2r)$ is known for $\alpha=1, 2, 4$. This suggests the following problem. *Find all values of r (if any) for which $N(\alpha t+\beta, 2r)$ is simple, where α is an arbitrary integer >0 and different from* 1, 2, 4. The obvious generalization in which the ζ, ξ functions are replaced by other given functions of divisors might also be considered.

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* Journal für Mathematik, vol. 35 (1847), p. 135.

NON-ISOLATED CRITICAL POINTS OF FUNCTIONS*

BY W. M. WHYBURN†

The isolated critical points of real functions of n independent real variables have been treated in an elegant manner by Marston Morse.‡ This treatment obtains definite relations between the numbers of critical points of $n+1$ types that appear in a bounded portion of the space of the independent variables. Morse requires his functions to have continuous third partial derivatives in the neighborhoods of the critical points and imposes conditions that are sufficient to insure the existence of at most a finite number of such points in the domain under consideration. In the present note§ we consider functions that have continuous first partial derivatives and may have an infinite number of critical points, or even continua of such points, in the given domain. As a special case of our results we obtain the *minimax principle* of Birkhoff in the modified form given by Bieberbach.‖

Let S denote the space of the n real variables $x_1, x_2, \cdots, x_n$, and let $(x)=(x_1, x_2, \cdots, x_n)$ denote a point in this space. Let R be a bounded,¶ open, and connected point set in S and let C, the boundary of R, be connected. Let the real function $f(x_1, \cdots, x_n)$ be single-valued and continuous

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* Presented to the Society, December 27, 1928.

† National Research Fellow in Mathematics.

‡ Transactions of this Society, vol. 27 (1925), pp. 345–396.

§ The author is indebted to Marston Morse for helpful suggestions concerning the material of this paper.

‖ *Differentialgleichungen*, Berlin, Springer, 1927, p. 140.

¶ The terms of classical point set theory are used in their usual sense. The distance between two points is given by a generalization of the ordinary distance formula in the plane. If K denotes a point set, then K' is used to denote K together with all of its limit points.

throughout R' and furthermore let $f_i(x_1, \cdots, x_n) = \partial f/\partial x_i$, $(i = 1, \cdots, n)$, be continuous in R'. On C, let $f(x) = K$, a constant, and at all points of $R = R' - C$ let $f(x)$ be less than K.

A point of R at which $f_1(x) = f_2(x) = \cdots = f_n(x) = 0$ is called a critical point of $f(x)$ and the value of $f(x)$ at such a point is called a critical value of $f(x)$. Let H denote the set of all critical points of $f(x)$ in R.

THEOREM 1. *For each point p of H there exists a connected subset $M(H, p)$ of H that contains p and has $f(x) = f(p)$ at each of its points and finally, such that $M(H, p)$ is not a proper subset of any other subset of H that is connected, contains p, and has $f(x) = f(p)$ at each of its points.*

PROOF. Let p be a point of H and let T denote the set of all points of H for which $f(x) = f(p)$. If T is connected, it is the desired set. Since T is bounded, we may find a hypercube C_1 of edge d containing T on its interior. Let $C_1, C_2, \cdots$ denote the set of hypercubes obtained by dividing C_1 into 2^n hypercubes $C_2, C_3, \cdots, C_{2^n+1}$ of edge $d/2$, then dividing each of these into hypercubes of side $d/4$ and continuing this subdivision indefinitely. We suppose these arranged so that for every i, C_i has its edge greater than or equal to the edge of C_{i+1}. Let C_{n_1} be the cube of lowest subscript that contains a point of T and is such that T can be expressed as the sum of two mutually exclusive closed sets P_1 and Q_1 such that P_1 contains p but no point of C'_{n_1}. Certainly such a cube exists since, by hypothesis, T is not connected and hence can be expressed as the sum of two mutually exclusive closed sets S_1 and S_2. There exists a positive number ϵ such that the sets S_1 and S_2 are at a distance apart greater than ϵ and hence any cube of the set $C_1, C_2, \cdots$, that has its edge of length less than $\epsilon/2$ and contains a point of the set which fails to contain p may be chosen. If P_1 is not connected, we let C_{n_2} be the hypercube of the set $C_1, C_2, \cdots$, of lowest subscript which contains points of P_1 and has the property that P_1 can be expressed as the sum of two mutually exclusive sets P_2 and Q_2 such that P_2 contains p but contains no point of C'_{n_2}.

This process may be continued indefinitely unless one of the P's, P_j, is connected. If P_j is connected, it is the desired set since $T-P_j=Q_1+Q_2+\cdots+Q_j$ and hence P_j cannot be a proper subset of a connected subset of T. If the process continues indefinitely, then the sequence $P_1, P_2, \cdots$, has at least one common point, namely, p. Let M denote the totality of such common points. M must be connected, since otherwise it would be the sum of two mutually exclusive closed sets (since M is closed). These sets would be at a distance apart greater than a definite positive constant and hence we could find a hypercube C_k of the set $C_1, \cdots$, containing points of that set which failed to contain p, but having no point of the other set in C_k'. This, however, is impossible since by our method of picking $C_{n_1}, \cdots, C_k$ would have been chosen at at least the kth stage. Since $T=M+\sum_{i=1}^{\infty} Q_i$, it follows that any point q of $T-M$ belongs to one of the Q's, Q_q, and since M belongs to P_q, it follows that q could not belong to a connected subset of T that contains M. Hence M is the desired set $M(H, p)$.

DEFINITION. *By the critical sets of $f(x)$ in R we mean the sets $M(H, p)$ of Theorem* 1.

THEOREM 2. *Let $K_1, K_2, \cdots, K_m$, where m is a positive integer, be subsets of H and let each K_i have the property that any two of its points can be joined in K_i by a rectifiable Jordan arc* of finite length. If $k=K_1'+\cdots+K_m'$ is connected, $f(x_1, \cdots, x_n)$ is constant on k.*

PROOF. A theorem on rectifiable curves in the plane and in three-space can readily be extended to hold in n-space.† This theorem states that if $x=f_1(t)$, $y=f_2(t)$, $z=f_3(t)$ is a rectifiable curve of finite length, then df_1/dt, df_2/dt, df_3/dt exist and are finite at every point of the t interval with the possible exception of a set of points of measure zero.

* The definition of such an arc is an obvious extension of the two-dimensional definition.

† Hobson, *Theory of Functions of a Real Variable*, Cambridge, 1921, vol. I, pp. 320 and 376.

Let p_1 and p_2 be any two points of K_i and let $x_j=f_j(t)$, $j=1, \cdots, n$, where t denotes the length of arc from p_1, be a Jordan arc of the type whose existence is conditioned in the theorem. We have $df=\sum_{j=1}^{n} f_j dx_j$ *almost everywhere* on the t interval and since the arc is a subset of H, we have $df=0$ *almost everywhere* on this arc. An application of Lebesgue integration yields $f(p_1)=f(p_2)$. Hence $f(x)$ is constant on K_i. From continuity, it is therefore constant on K_i' and from the connectedness of k it follows that $f(x)$ has a fixed constant value on this set of points.

DEFINITION. *A critical set $M(H, p)$ is said to be a minimal set of $f(x)$ provided it is true that for every point q of $M(H, p)$ there exists a positive number ϵ_q such that $f(x) \geqq f(q)$ for every point (x) whose distance from q is less than ϵ_q, the equality sign holding only for points of $M(H, p)$.*

THEOREM 3. *If $f(x_1, \cdots, x_n)$ has two distinct minimal sets M_1 and M_2 in R, there exists at least one critical point of $f(x_1, \cdots, x_n)$ in R that does not belong to M_1+M_2.*

PROOF. Let the notation be chosen so that if m_1 and m are the values of $f(x)$ on M_1 and M_2, respectively, then $m \geqq m_1$. Let the notation $G(f \leqq h)$ mean the subset of R' on which $f \leqq h$, and let $G(f=h)$ mean the subset of R' on which $f=h$, while $G(f<h)$ means the subset of R' on which $f<h$. In $G(f \leqq m)$ there are two closed, connected, and mutually exclusive sets N_1 and N_2 such that N_1 contains M_1, $N_2 \equiv M_2$, and N_1 and N_2 are the maximal connected subsets* of $G(f \leqq m)$ that contain M_1 and M_2, respectively. That N_1 and N_2 are mutually exclusive follows when one notes that if they had a point q in common it would follow from the connectedness of N_1 and N_2 that this point would belong to M_2 and be a limit point of points of $N_1 - M_1$ (since M_1 is closed), that is, points

* By a maximal connected subset of a set T having certain properties we understand a subset which has the properties and is connected but which is not a proper subset of any other subset of T having these properties. The existence of such sets for each of our applications is established by a repetition of the arguments used in proving Theorem 1 with slight verbal modifications.

of $G(f<m)$. This is impossible since M_2 is a minimal set of $f(x)$.

Let h be any number such that $m \leqq h \leqq K$ and let N_{1h} and N_{2h} be the maximal connected subsets of $G(f \leqq h)$ that contain M_1 and M_2, respectively. For $h=m$, N_{1h} and N_{2h} have no point in common while for $h=K$, these two sets have a common point and are therefore identical. An application of the Dedekind-cut postulate yields a number k, $m \leqq k \leqq K$, such that k is either the greatest number for which N_{1k} and N_{2k} have no point in common or it is the least number for which $N_{1k} \equiv N_{2k}$. We show that k is the least number for which $N_{1k} \equiv N_{2k}$. Suppose k is the greatest number for which N_{1k} and N_{2k} are mutually exclusive. Noting that $R'=G(f \leqq K)$ is connected, we can and will choose a sequence $k_1 > k_2 > \cdots > k_i > \cdots$, so that $k<k_i<K$ and $\lim_{i\to\infty} k_i = k$. $N_i = N_{1k_i} = N_{2k_i}$, $i=1, 2, \cdots$, is a sequence of closed, bounded, and connected sets having the property that for each i, N_{i+1} is a subset of N_i. A direct generalization of the usual proof in the plane, making use of the Heine-Borel property and the set of hypercubes introduced in the proof of Theorem 1, shows that the common part, $N_{1k}+N_{2k}$, of this sequence is closed and connected. This, however, contradicts the hypothesis that k was a number for which the sets N_{1k} and N_{2k} have no common point and shows that k is the least number for which $N_{1k} \equiv N_{2k}$.

The set of points $U=G(f=k)$ contains no point of M_1+M_2. Assume that U contains no critical point of $f(x)$. We show that this assumption leads to a contradiction.

LEMMA 1. *U consists of at most a finite number of maximal connected subsets.*

PROOF. Assume the contrary; then $U=u_1+u_2+\cdots$, where no two of the u_i's have a point in common, each u_i is closed, and the u_i's are chosen so that the sum of two or more of them is not connected. There is at least one point p of U that is a limit point of a sequence of points $p_1, p_2, \cdots$ of U and no two of the p_j's belong to the same u_i. Since one of the partial derivatives, f_j, is different from zero at p and

f_j is continuous in R, we may find a neighborhood D of p throughout which f_j is different from zero. Furthermore, by the classical implicit function theorem* there exists a subneighborhood E of D such that all points of U that lie in E belong to a single continuous function $x_j = F(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$ and hence the subset of U that lies in E is connected. This contradicts the hypothesis that p is a limit point of points of infinitely many distinct u_i's. Hence $U = U_1 + U_2 + \cdots + U_r$, where r is a positive integer and U_i, $i = 1, \cdots, r$, is closed and connected.

LEMMA 2. *U_i, $i = 1, \cdots, r$, is a closed manifold† in the sense of analysis situs and its connectivity R_{n-1} is given by $R_{n-1} = 2$.*

PROOF. S. S. Cairns‡ has recently shown that U_i is a complex in the sense of analysis situs. This is accomplished by breaking up U_i into $(n-1)$-cells. An application of the implicit function theorem used in proving Lemma 1 shows that U_i has each of its $(n-2)$-cells incident with exactly two $(n-1)$-cells and hence U_i has no boundary. Hence U_i is a closed manifold and since it consists of a single $(n-1)$-circuit and this is, of course, non-bounding, we have $R_{n-1} - 1 = 1$, or $R_{n-1} = 2$.

PROOF OF THEOREM 3. By Alexander's theorem§ $S - U_i$, $(i = 1, \cdots, r)$, consists of two mutually exclusive connected point sets C_i and D_i. More generally, this theorem shows that $S - \sum_{n=1}^{j} U_i$ consists of $j+1$ mutually exclusive connected sets $G_1, \cdots, G_{j+1}$. Since $S - \sum_{i=1}^{r} U_i$ consists of two mutually exclusive sets one of which contains M_1 and the other contains M_2, it follows as an immediate consequence of a theorem by Alexandroff‖ that at least one of the sets U_i, call it U_1, is such that $S - U_1 = C_1 + D_1$, where C_1 contains

* See Hobson, loc. cit., p. 410.

† See, J. W. Alexander, Transactions of this Society, vol. 23 (1922), pp. 333 ff.

‡ Unpublished work. An indication of the procedure is given by Morse, loc. cit., p. 355.

§ Loc. cit., p. 343, Theorem Y.

‖ Comptes Rendus, vol. 183, p. 723, Theorem II_n.

M_1 and D_1 contains M_2. Since C is connected, it is a subset of C_1 or D_1. A reapplication of Alexandroff's theorem yields U_2 such that $S-(U_1+U_2)=G_1+G_2+G_3$, where G_1 contains M_1, G_2 contains M_2, and G_3 contains C. For definiteness let D_1 contain M_2 and C. We distinguish between two cases: Case I, C_2 contains M_1, M_2, and U_1 while D_2 of necessity contains C; Case II, C_2 contains M_1, U_1, and C while D_2 contains M_2. No other cases can arise.

CASE 1. Since C_2 contains U_1 and since $f(x)$ is continuous over C_2+U_2, it follows that $f(x)$ assumes its maximum for this domain in at least one point of the domain. If this maximum is greater than k then the point is an interior point and a consideration of the difference quotients of which the partial derivatives of $f(x)$ are the limits shows immediately that this point is a critical point and thereby contradicts our hypothesis. If the maximum has the value k, a similar consideration of the difference quotients shows that every point of U_1 is a critical point and likewise contradicts our hypothesis.

CASE 2. In this case C_1+U_1 and D_2+U_2 are closed and mutually exclusive. $N_{1k}=[C_1+U_1]+[D_2+U_2]+Q$, where Q cannot be vacuous since N_{1k} is connected. If A is a point of Q, we find U_3 so that $S-U_3=C_3+D_3$, where C_3 contains A and D_3 contains C. If either M_1 or M_2 belongs to C_3, a repetition of the argument of Case 1 proves the existence of a critical point of the type desired for Theorem 3. Hence both C_1 and D_2 are subsets of D_3 and hence neither they nor their boundaries can contain limit points of the subset of Q that belongs to C_3. A repetition of this argument a finite number of times (at most r times) together with the observation that $\sum_{i=3}^{r} U_i$ is closed shows that $[C_1+U_1]$ and $[D_2+U_2]$ contain no limit points of Q. Hence N_{1k} cannot be connected, contrary to the manner in which it was constructed. This contradiction yields Theorem 3.

COROLLARY 1.* *If $f(x_1, \cdots, x_n)$ has two distinct minimal*

* This corollary follows directly from the proof of Theorem 3.

sets M_1 and M_2 in R, there exists at least one critical point of $f(x_1, \cdots, x_n)$ in R that does not belong to a minimal set of $f(x_1, \cdots, x_n)$.

COROLLARY 2. *If $f(x_1, \cdots, x_n)$ is restricted so that each of its minimal sets contains a single point, Theorem 3 becomes Bieberbach's form of the minimax principle.**

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SYMMETRIC FUNCTIONS OF n-IC RESIDUES (mod p)†

BY T. A. PIERCE

If p be an odd prime, q is said to be an n-ic residue of p if the congruence $x^n \equiv q \pmod{p}$ has solutions; otherwise q is an n-ic non-residue of p. A necessary and sufficient condition that q be an n-ic residue of p is that

$$q^{(p-1)/\delta} \equiv 1, \pmod{p}, \tag{1}$$

where $\delta = \text{g.c.d. } (p-1, n)$. The number‡ of n-ic residues of a given prime p is $(p-1)/\delta$.

It is with the symmetric functions of these n-ic residues that this paper deals.

By means of (1) we readily prove that the product of two n-ic residues is an n-ic residue and that the product of an n-ic residue by an n-ic non-residue is an n-ic non-residue.

Put $(p-1)/\delta = r$ and let $q_1, q_2, \cdots, q_r$ be the set of all distinct n-ic residues of p. Then $q_1q_1, q_1q_2, \cdots, q_1q_r$ is the same set in different order, for the assumption that two members of this last set are congruent leads to the conclusion that two members of the first set are not distinct.

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* Loc. cit., p. 140.

† Presented to the Society, March 30, 1929.

‡ Dirichlet-Dedekind, *Zahlentheorie*, 4th ed., 1894, p. 74.

Let $S_\nu(q_1, q_2, \cdots, q_r)$ be an integral homogeneous symmetric function of $q_1, q_2, \cdots, q_r$ of total degree ν. Assume that ν is not a multiple of r. Take q_i to be an n-ic residue belonging to the exponent r; such a q_i exists since $\phi(r)>1$. Then by virtue of the homogeneity

$$S_\nu(q_iq_1, q_iq_2, \cdots, q_iq_r) = q_i^\nu S_\nu(q_1, q_2, \cdots, q_r).$$

Moreover

$$S_\nu(q_iq_1, q_iq_2, \cdots, q_iq_r) \equiv S_\nu(q_1, q_2, \cdots, q_r), \pmod{p}.$$

Hence

$$(q_i^\nu - 1)S_\nu(q_1, q_2, \cdots, q_r) \equiv 0, \pmod{p}.$$

Since q_i belongs to the exponent r and ν is not a multiple of r, the first factor is not divisible by p; therefore the second factor is divisible by p. Hence we may state the following theorem.

THEOREM 1. *An integral symmetric function of dimension ν of the n-ic residues of the odd prime p is divisible by p if ν is not a multiple of $(p-1)/\delta$.*

Any symmetric function of the rth powers of the q's may be evaluated by (1). Thus

$$\sum q_i^r \equiv 1 + 1 + \cdots = (p-1)/\delta \equiv -\delta^{p-2}, \pmod{p}.$$

A general theory of symmetric functions whose dimensions are multiples of $(p-1)/\delta = r$ is a desideratum here as it is in the case of ordinary residues when the dimensions are multiples of $p-1$. The two simplest cases will be treated here.

First, since

$$x^{(p-1)/\delta} - 1 \equiv 0, \pmod{p},$$

is satisfied by the $(p-1)/\delta$ distinct n-ic residues this congruence has its full complement of real roots and hence

$$q_1q_2 \cdots q_r \equiv \pm 1, \pmod{p}, \tag{2}$$

according as r is odd or even.

Next assume that $(p-1)/\delta$ is even and greater than 2 and let q belong to the exponent $r/2>1$. Such a q exists since $\phi(r/2)>1$. Then $(q^{r/2})^{\delta}\equiv q^{(p-1)/2}\equiv 1 \pmod{p}$ and q is a quadratic residue of p. Thus the congruence

$$x^2 - q \equiv 0, \pmod{p} \tag{3}$$

has a solution x. This solution x must be an n-ic residue, for from $x^2\equiv q \pmod{p}$ we have $x^r\equiv q^{r/2}\equiv 1 \pmod{p}$, and x satisfies (1).

The fact that (3) has a solution among the set of q's may be expressed

$$(q - q_1^2)(q - q_2^2)\cdots(q - q_r^2) \equiv 0, \pmod{p}.$$

Expanding the left member and omitting intermediate terms whose coefficients are multiples of p, by the theorem above we have

$$q^r + (-1)^{r/2}S_{r/2}(q_1^2, \cdots, q_r^2)q^{r/2} + (q_1\cdots q_r)^2 \equiv 0, \pmod{p}, \tag{4}$$

where $S_{r/2}(q_1^2, \cdots, q_r^2)$ is the sum of the products of $q_1^2, \cdots, q_r^2$ taken $r/2$ at a time. The dimension of $S_{r/2}$ is thus $(p-1)/\delta$. The last term of (4) is unity by (2) and since q belonged to the exponent $r/2$ we have

$$S_{r/2}(q_1^2, \cdots, q_r^2) \equiv (-1)^{r/2+1}\,2, \pmod{p}.$$

This last result corresponds to one due to Glaisher* for ordinary residues.

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* Glaisher, *Congruences relating to sums of products*, etc., Quarterly Journal of Mathematics, vol. 31 (1900), p. 34.

SINGULAR MANIFOLDS AMONG THOSE OF AN ANALYTIC FAMILY*

BY O. D. KELLOGG

1. *Exceptional Occurrence of Singular Manifolds.* This note is concerned with the following theorem.

Let R denote a closed region, consisting of an open continuum of the space of the n complex variables $z_1, z_2, \cdots, z_n$, *together with its boundary points. Let the functions* $F_i(z_1, z_2, \cdots, z_n)$, $i=1, 2, \cdots, m$, $m \leqq n$, *be analytic at all points of R. Let* M_k *denote the matrix*

$$(M_k) \qquad \left\| \frac{\partial F_i}{\partial z_j} \right\|, \qquad \begin{matrix} (i = 1,2, \cdots, k), \\ (j = 1,2, \cdots, n). \end{matrix}$$

We assume that M_m *is of rank m at some point of R.*

Consider the manifold defined by the equations

$$(A) \quad F_1(z_1,z_2, \cdots, z_n) = c_1, \cdots, F_m(z_1,z_2, \cdots, z_n) = c_m,$$

where $c_1, c_2, \cdots, c_m$ *are complex constants.*

For all but a finite number of values of c_1 *the manifold defined by the first equation (A) contains no points in R at which the rank of* M_1 *is less than* 1.

If $c_1, c_2, \cdots, c_k$ *have been chosen so that the matrix* M_k *is of rank k at every point in R on the manifold defined by the first k equations (A), then for all but a finite number of values of* c_{k+1} *the manifold defined by the first* $k+1$ *equations (A) contains no points in R at which the rank of the matrix* M_{k+1} *is less then* $k+1$.

Thus, if $c_1, c_2, \cdots, c_m$ *are chosen in order, each avoiding a certain finite set of values, the manifold defined by the equations (A) will have no singular points in R.*†

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* Presented to the Society, March 29, 1929.

† As far as I know, a proof of this general theorem has not been published. Birkhoff and I gave it for the case in which the functions F_i are polynomials (Transactions of this Society, vol. 23 (1922), pp. 97–98), and

By a singular point of a manifold, we mean a point at which a unique tangent linear manifold of the same order does not exist. A sufficient condition that a point of the manifold defined by the equations (A) be not singular is that the rank of the matrix M_m is m at that point. That the condition is not necessary may be seen by the example

$$z_1^2 z_2^2 - z_3^3 = 0, \qquad z_1^2 z_2^2 + z_3^3 = 0.$$

The manifold defined by these equations consists of the two intersecting straight lines $z_1 = z_3 = 0$ and $z_2 = z_3 = 0$. Its only singular point is at the origin, yet the rank of M_2 is less than 2 at every point of the manifold.

It is not stated in the theorem that the values of the c_j characterizing singular manifolds are finite in number. In general, the value of c_{k+1} to be avoided, if the resulting manifold is to be non-singular, is a function of the c_j previously chosen. Thus in the case

$$z_1 - z_2 z_3 = c_1, \qquad z_1 = c_2,$$

the value of c_2 to be avoided is c_1. What is the case, is that among the ∞^{2m} members of the family defined by (A), there are at most ∞^{2m-2} singular manifolds, a precise interpretation of this statement being given in the theorem.

2. *Needed Results in the Theory of Analytic Functions.* We attach our considerations to the study of simultaneous equations in the second volume of Osgood's *Lehrbuch der Funktionentheorie*,* where we find a theorem, of which the part we need may be stated as follows:

Given a system of simultaneous equations

$$G_1(z_1, z_2, \cdots, z_n) = 0, \cdots, G_l(z_1, z_2, \cdots, z_n) = 0,$$

where $G_k(z_1, z_2, \cdots, z_n)$, $k = 1, 2, \cdots, l$, *is analytic at the point* $(a) = (a_1, a_2, \cdots, a_n)$ *and vanishes there, but does not*

Morse gives a proof for the case $m = 1$ in a paper in the April number of the American Journal of Mathematics, vol. 51 (1929). The theorem is apparently useful; in addition to its role in the two papers just cited it has served me at several points in a forthcoming book on Potential Theory.

* Leipzig, 1st ed., 1924, 2d ed., 1929, Chap. II, §17.

vanish identically. The solutions of this system in the neighborhood of (a) *consist either of the point* (a) *alone, or of one or more manifolds* g. The context makes it clear that the number of these manifolds is *finite.*

The essential property of the manifolds g for our purposes is that *any two points of any one of them can be connected by a curve* γ, *lying in the manifold, and analytic except at a finite number of points.*

We begin, in establishing this property, with a pseudo-algebraic manifold $\mathfrak{G}$ defined by the irreducible equation

$$(1)\quad H(w, u_1, u_2, \cdots, u_\rho) = w^\mu + E_1 w^{\mu-1} + \cdots + E_\mu = 0$$

and the inequalities $|u_i| < h$, where h is a suitably restricted positive number. The coefficients E_j are power series in the variables u_i convergent for $|u_i| < h'$, $h' > h$, vanishing at the origin $O(u_1 = u_2 = \cdots = u_\rho = 0)$.

If $P'(w', u_1', u_2', \cdots, u_\rho')$ is any point of $\mathfrak{G}$, and if in the equation (1) we set $u_1 = u_1' t$, $u_2 = u_2' t$, $\cdots$, $u_\rho = u_\rho' t$, the resulting equation defines w as an algebroid function of t. If the latter equation is reducible, we select the irreducible factor of the left hand member which vanishes for $w = w'$ when $t = 1$, and equate it to 0. The discriminant of the resulting equation vanishes at most a finite number of times on the real interval (0, 1), and hence from the branches of the function w of t there can be selected a (complex) single-valued function $w = f(t)$, which is continuous on the closed interval (0, 1), analytic except at a finite number of points, and which reduces to 0 for $t = 0$, and to w' for $t = 1$. Then the curve

$$(\gamma')\qquad w = f(t), \quad u_1 = u_1' t, \cdots, \quad u_\rho = u_\rho' t, \quad 0 \leqq t \leqq 1,$$

lies in $\mathfrak{G}$, connects P' with O, and is analytic except at a finite number of points. By means of a curve consisting of two such parts, any two points P' and P'' of $\mathfrak{G}$ can be connected.

We now consider a set of functions $w_1, w_2, \cdots, w_\sigma$ of

$u_1, u_2, \cdots, u_\rho$, belonging to the pseudo-algebraic manifold $\mathfrak{G}$, that is, having the following properties: they are one-valued and continuous on $\mathfrak{G}$, and analytic at every point of $\mathfrak{G}$ at which the discriminant of the equation (1) does not vanish; the function $\alpha_1 w_1 + \cdots + \alpha_\sigma w_\sigma$ of the u_i has, for non-specialized values of the parameters $\alpha_1, \cdots, \alpha_\sigma$, branches, corresponding to the branches of w, no two of which are identically equal. Then the set of points of the space of $(w_1, w_2, \cdots, w_\sigma, u_1, u_2, \cdots, u_\rho)$ which correspond to the points $(w, u_1, \cdots, u_\rho)$ of $\mathfrak{G}$ constitutes a manifold g. The most general manifold g is either one defined in this way, or else one derived from it by a non-singular linear transformation of the variables. As such a transformation does not change the essential character of the curve γ, we may confine ourselves to the manifolds g as first defined.

The functions $w_1, w_2, \cdots, w_\sigma$ are pseudo-algebraic, and as they are one-valued and continuous on $\mathfrak{G}$, they assume at O unique values $w_{10}, w_{20}, \cdots, w_{\sigma 0}$. If $p'(w_1', w_2', \cdots, w_\sigma', u_1', u_2', \cdots, u_\rho')$ is any point of g, there corresponds to it a unique point P' of $\mathfrak{G}$, which is connected with O by a curve γ'. Just as the function $w = f(t)$ was determined, we may determine from among the branches of w_1, a single-valued function $w_1 = f_1(t)$, continuous on the interval $(0, 1)$, analytic, except at a finite number of points, and reducing to w_{10} for $t = 0$ and to w_1' for $t = 1$. When this and similar functions have been determined for $w_1, \cdots, w_\sigma$, we shall have a curve

$$(\gamma)\quad w_1 = f_1(t), \cdots, w_\sigma = f_\sigma(t), u_1 = u_1' t, \cdots, u_\rho = u_\rho' t, \qquad 0 \leqq t \leqq 1,$$

from two of which can be constructed a curve of the required character, connecting any two given points of g.

3. *Proof of the Theorem.* Let (a) be any point of R. Then in the neighborhood of (a), the solutions of the equations

$$(2)\qquad \frac{\partial F_1}{\partial z_1} = 0, \quad \frac{\partial F_1}{\partial z_2} = 0, \cdots, \frac{\partial F_1}{\partial z_n} = 0,$$

(not all of which are identically satisfied, since M_m is of rank m at some point of R), if there are any, consist, by Osgood's theorem, either of the point (a) alone, or of a finite number of manifolds g. Along an analytic arc of a curve γ joining any two points of such a manifold, $dF_1=0$. It follows that F_1 is constant on each such manifold, so that the number of values of F_1 corresponding to solutions of the equations (2) in the neighborhood of (a), is finite. By the Heine-Borel theorem, the same is true for the whole region R. Hence for all but a finite number of values of c_1 the manifold defined by the first equation (A) contains no points in R at which M_1 is of rank less than 1. The first part of the theorem is thus established.

Suppose now that $c_1, c_2, \cdots, c_k$ have been chosen so that at every point in R of the manifold defined by the first k equations (A), the matrix M_k is of rank k. We consider the simultaneous system of equations consisting of the two sets

$$(3)\ F_1(z_1,z_2,\cdots,z_n)-c_1=0,\ \cdots,F_k(z_1,z_2,\cdots,z_n)-c_k=0,$$

and

$$(4)\ D_1(z_1,z_2,\cdots,z_n)=0,\ \cdots,\ D_r(z_1,z_2,\cdots,z_n)=0,$$

where $D_1, \cdots, D_r$ are all the determinants of order $k+1$ formed from the matrix M_{k+1}, except such as may be identically 0. Not all vanish identically, since, by hypothesis, M_m is of rank m at some point of R.

If this system has any solutions in the neighborhood of a point (a) of R, they consist, by Osgood's theorem, of the point (a) alone, or of a finite number of manifolds g. Let g denote such a manifold. On it M_k and M_{k+1} are everywhere of rank k. Hence from the equations $dF_1=dF_2=\cdots=dF_k=0$, which follow from (3), we infer that $dF_{k+1}=0$ at all points of the curve γ connecting any two given points of g. By reasoning previously employed, it follows that for all but a finite number of values of c_{k+1}, the matrix M_{k+1} is of rank $k+1$ at all points of R on the manifold defined by the first $k+1$ equations (A). This proves the second part of the theorem.

The truth of the final statement of the theorem emerges when to k are assigned successively the values $1, 2, \cdots, m-1$, and when it is recalled that a sufficient condition that the manifold defined by the equations (A) has no singular points in R is that the matrix M_m is of rank m at all points of the manifold in R.

HARVARD UNIVERSITY

ON THE FOUNDATIONS OF GENERAL INFINITESIMAL GEOMETRY*

BY HERMANN WEYL

In connection with a seminar on infinitesimal geometry in Princeton, in which I took part, it seemed desirable to clarify the relations between the work of the Princeton school and that of Cartan.

With a group $\mathfrak{G}$ of transformations in m variables ξ is associated, in accordance with Klein's Erlanger Program, a homogeneous or plane space $\mathfrak{R}$ of the kind $\mathfrak{G}$; a point of $\mathfrak{R}$ is represented by a set of values of the "coordinates" ξ^α and figures which go into each other on subjecting the coordinates to a transformation of $\mathfrak{G}$ are to be considered as fully equivalent. The transformations of $\mathfrak{G}$ give at the same time the transition between two allowable "normal" coordinate systems in $\mathfrak{R}$. If we have two spaces $\mathfrak{R}$, $\mathfrak{R}'$ of the kind $\mathfrak{G}$ and set up a definite normal coordinate system in each of them, then such a transformation can be interpreted as an isomorphic representation of $\mathfrak{R}$ on $\mathfrak{R}'$. $\mathfrak{G}$ is assumed to be transitive.

Cartan† developed a general scheme of infinitesimal geometry in which Klein's notions were applied to the tangent plane and not to the n-dimensional manifold M itself. The

* Presented to the Society, June 21, 1929

† E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée*, Annales de l'École Normal Supérieure, vol. 40 (1923), pp. 325–412; in particular, p. 383, etc.

object of his work can be briefly described as follows. There is associated with each point P of M an m-dimensional plane space of the kind $\mathfrak{G}$. To the transition from P to a neighboring point P', that is, to the line element PP', corresponds an isomorphic representation of T_P on $T_{P'}$, the "displacement" $T_P \rightarrow T_{P'}$ or briefly $\overline{PP'}$. On this basis the introduction of a general concept of curvature is possible: if we displace the tangent plane T_P in P along a curve L on M which leads back to P the tangent plane returns in a new position or orientation. The final position is obtained from the original by a certain isomorphic representation of T_P onto itself, and this we call the "curvature along L."

For the purpose of analytic formulation we refer M to coordinates x^i and introduce a normal coordinate system ξ^α in each tangent plane T_P. Let the components of $\overline{PP'}$ be dx^i. We assume that the ξ^α of all the T_P can be so chosen that the displacement PP' is an infinitesimal isomorphic representation of the same order of magnitude as the dx^i. A radical specialization is introduced by the further assumption that this displacement depends linearly on $\overline{PP'}$, i.e. that the consecutive application of the displacements PP' and $P'P''$ shall yield the displacement PP''. The displacement from P to all neighboring points P' is then expressed by the formula

$$d\xi^\alpha = -\sum_i u_i^\alpha(\xi)dx^i. \tag{1}$$

The curvature along the infinitesimal parallelogram formed by line elements dx and δx, which consequently has as components

$$(\Delta x)^{ik} = dx^i\delta x^k - \delta x^i dx^k,$$

is given by

$$\Delta\xi^\alpha = R_{ik}^\alpha(\xi)dx^i\delta x^k = \tfrac{1}{2}R_{ik}^\alpha(\xi)(\Delta x)^{ik},$$

$$R_{ik}^\alpha = \left(\frac{\partial u_k^\alpha}{\partial x^i} - \frac{\partial u_i^\alpha}{\partial x^k}\right) + \left(\frac{\partial u_k^\alpha}{\partial \xi^\rho}u_i^\rho - \frac{\partial u_i^\alpha}{\partial \xi^\rho}u_k^\rho\right).$$

In the foregoing the tangent plane is not tied up with the manifold; in order to justify this designation and hold to the

idea of a tangent plane we must now imbed it into the manifold. The first step in this process consists in taking a definite point O of T_P as center which shall, by definition, cover the point P on M (imbedment of 0th order). This leads to a restriction in the choice of a normal coordinate system ξ on T_P; because of the transitivity of $\mathfrak{G}$ it can and shall be so chosen that the coordinates ξ vanish in the center. The group $\mathfrak{G}$ is restricted to the subgroup $\mathfrak{G}_0$ of all representations of $\mathfrak{G}$ which leave the center O invariant. On displacing the tangent plane along a closed curve L, O goes over into a point O^* whose deviation from O characterizes the "torsion along L." $R^\alpha_{ik}(0)=0$ is the necessary and sufficient condition that M be without torsion.

The idea of tangent plane further requires that the line elements of T_P issuing from O shall "coincide" with the line elements of M issuing from P; this correspondence must be a one-to-one affine representation. But having already required imbedment of 0th order the method of accomplishing this imbedment of 1st order is fixed. The center O' of $T_{P'}$ arises by the displacement PP' from a definite point O_1 of T_P, and we let the line element $\overline{OO_1}$ on T_P correspond to the line element $\overline{PP'}$ on M. For purposes of calculation it is, however, more convenient to consider the line element $\overline{O_1'O'}$ on $T_{P'}$ which arises from $\overline{OO_1}$ by the displacement PP'. The ξ-coordinates of O_1' on $T_{P'}$ are $d\xi^\alpha=-u_i^\alpha(0)dx^i$, and consequently

$$d\xi^\alpha = u_i^\alpha(0)dx^i \tag{2}$$

are the components of the line element $\overline{O_1'O_1}'$ on $T_{P'}$ or $\overline{OO_1}$ on T_P. The condition that this linear relation between (dx) and $(d\xi)$ be one-to-one reciprocal involves two requirements: (1) the dimensionality m of the tangent plane (which was until now arbitrary) must be the same as the dimensionality n of the manifold M, and (2) the determinant $|u_i^\alpha(0)|\neq 0$. If $\mathfrak{G}$ contains the affine group, and we shall henceforth assume that it does, the coordinate system ξ^α on T_P can be further adapted to the given coordinate system x^i on M in such a

way that corresponding line elements shall have the same components: $u_i^\alpha(0) = \delta_i^\alpha$.

If $\mathfrak{G}$ were the affine group the previous requirements would fully specify the normal coordinate system ξ^α on T_P in its dependence on the coordinates x^i chosen on M; but this is not the case if $\mathfrak{G}$ is a more extensive group. That is, the "tangent plane" T_P is not as yet uniquely determined by the nature of M, and so long as this is not accomplished we can not say that Cartan's theory deals only with the manifold M. Conversely, the tangent plane in P in the ordinary sense, that is, the linear manifold of line elements in P, is a centered affine space; its group $\mathfrak{G}$ is not a matter of convention. This has always appeared to me to be a deficiency of the theory; I consider that above all, the infinitesimal-geometric researches of Eisenhart, Veblen, T. Y. Thomas, and others in Princeton† have remedied this blemish for projective and conformal geometry.

The connection between ξ and x, although not yet uniquely determined by the previous postulates, allows us to conclude the following: In the development

$$R_{ik}^{\alpha}(\xi) = R_{ik}^{\alpha}(0) + R_{\beta ik}^{\alpha}\xi^\beta + \cdots,$$

the quantities $R_{ik}^{\alpha}(0)$ and $R_{\beta ik}^{\alpha}$ in the point P are determined by the coordinates x^i alone and transform on transformation of coordinates as tensors of order 3 and 4, of the kind indicated by the position of the indices. It is therefore an invariantive restriction to require that our manifold be such that (1) it is without torsion and (2) $\sum_\alpha R_{\alpha ik}^{\alpha}$ vanish; we call such a manifold "special."

Let $\mathfrak{G}$ be the projective group. We must then proceed to imbedment of second order in order that T_P be completely

† L. P. Eisenhart, *Non-Riemannian Geometry*, 1927. O. Veblen, *Projective tensors and connections*, Proceedings of the National Academy, vol. 14 (1928), p. 154; *Conformal tensors and connections*, ibid., vol. 14 (1928), p. 735. T. Y. Thomas, *A projective theory of affinely connected manifolds*, Mathematische Zeitschrift, vol. 25 (1926), p. 723.

determined by M. We consider, as an analog, the contact of two surfaces

$$y = f(x^1, \cdots, x^n), \quad y = f'(x^1, \cdots, x^n)$$

in $(n+1)$-dimensional space. Let

$$f' - f = a + a_i x^i + \tfrac{1}{2} a_{ik} x^i x^k + \cdots$$

in the neighborhood of the origin. There is contact of 0th order (intersection) if $a=0$, of 1st order (tangency) if in addition the linear terms are not present, $a_i=0$, and finally contact of second order (osculation) if further all a_{ik} vanish. I refer to the two surfaces as semi-osculating if in addition to a and a_i the sum $\sum_{(i)} a_{ii}$, the spur of the quadratic terms, vanishes. Analogously we demand that T_P not only be tangent to the manifold but further that it be semi-osculating. The name tangent plane is then misleading, but we shall use it instead of the more correct "projective semi-osculating plane" for the sake of brevity. The exact definition is as follows. Given an infinitesimal volume element V in P, say a parallelepipedon obtained from line elements in P which shall be infinitely small in comparison with $\overline{PP'} = (dx^i)$. Let V' be the "same" volume element in P', that is, it shall be generated from the line elements with the same components in P'; naturally this construction is dependent on the particular coordinate system employed. Because of the imbedment of order V coincides with a volume element V in O on T_P and V' with one such in O' on $T_{P'}$; this latter is obtained from an element V_1 in O_1 on T_P by the displacement PP'. We now require that V and V_1 have the same volume, measured in the coordinates ξ on T_P.

It is again more convenient for the calculation to take V and V_1 over into V_1', V' on $T_{P'}$ by means of the displacement PP'. The isomorphic representation which carries V' into V_1' is by definition simply (1)—taken for ξ's which are infinitely small compared to the dx^i. Consequently we write

$$u_i^{\alpha}(\xi) = \delta_i^{\alpha} + \Gamma_{\beta i}^{\alpha} \xi^{\beta}.$$

and on introducing

$$\Gamma^{\alpha}_{\beta i} dx^i = d\gamma^{\alpha}_{\beta},$$

we find

$$\log (V'/V_1') = \log (V_1/V) = \sum_{\rho} d\gamma^{\rho}_{\rho}.$$

Our condition is that this trace shall vanish, and we assert that it can be fulfilled by appropriate choice of the projective coordinates ξ^α on T_P. The previous requirements determine the ξ except for a projective transformation of the type

$$\bar{\xi}^i = \frac{\xi^i}{1 + \sum \alpha_k \xi^k},$$

which leaves the center unaltered and is the identity to terms of first order in the neighborhood of the center. The ratio of the measures of volume elements V and $\overline{V}$ in ξ and $\bar{\xi}$, situated at (ξ^α), is given by the functional determinant

$$\left| \frac{\partial \bar{\xi}^i}{\partial \xi^j} \right| = (\bar{\xi}\xi).$$

For infinitely small ξ this determinant is

$$(\bar{\xi}\xi) = 1 - (n+1) \sum \alpha_k \xi^k.$$

Our volume V is at the point $\xi = 0$, V_1 at $\xi^\alpha = dx^\alpha$, and consequently

$$\log (\overline{V}_1/\overline{V}) = \log (V_1/V) - (n+1) \sum \alpha_i dx^i$$

from which we see that in order that $\overline{V}_1 = \overline{V}$ the α_i can be chosen in one and only one way; $\alpha_i = \Gamma^{\rho}_{\rho i}$.

The projective coordinate system ξ on T_P is now completely specified by the coordinates x^i on M. If we refer M to a new coordinate system $\bar{x}^i$ we shall have a new projective coordinate system ξ on T_P. The projective transformation $\xi \to \bar{\xi}$ can be described by the facts (1) that it agrees in the neighborhood of the origin with the transformation $x \to \bar{x}$ in terms of first order about P and (2) that the functional determinant $(\bar{\xi}\xi)$ in 0 agrees with $(\bar{x}x)$ in terms of 1st as well as

0th, order. H. P. Robertson* pointed out in a short note that this relation is the decisive point in Veblen's transformation theory of projective space. What we do here, however, is not simply connect the *transformations* of the ξ with those of x but rather we associate a projective ξ *coordinate system* on T_P with an individual x system on M. This possibility arises from the fact that we begin with the projective connection and with its aid tie up the ξ with the x, i.e. accomplish the complete imbedment of the tangent plane. But on the other hand the transformation of the ξ is determined by the transformation of the x, as described above, without taking the given projective connection into account. Veblen's procedure corresponds to this method: this relation between the two transformations is first obtained and the corresponding invariant theory of possible projective connections then developed.

The introduction of $n+1$ homogeneous projective coordinates η by means of the equation $\xi^i = \eta^i/\eta^0$ is for the present purely a matter of convenience. The formulas for the displacement expressed in terms of them have the form

$$d\eta^\alpha = -\, d\gamma_\beta^\alpha \eta^\beta, \quad d\gamma_\beta^\alpha = \Gamma_{\beta i}^\alpha dx^i. \tag{3}$$

(From now on Greek indices shall run from 0 to n and Latin from 1 to n.) Since only the ratios of the η are to be considered we can and shall introduce the normalizing condition $d\gamma_0^0 = 0$. We then have $\Gamma_{\rho i}^\rho = \delta_i^\rho$ and $\Gamma_{\rho i}^\rho = 0$. In the case of a special manifold (see above) we have furthermore the symmetry condition $\Gamma_{ki}^\sigma = \Gamma_{ik}^\sigma$ and the Γ with only Latin indices determine the remaining components. This leads to the theorem: *If we allow only special manifolds the projective connection is determined uniquely by the geodesics.*†

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* H. P. Robertson, *Note on projective coordinates*, Proceedings of the National Academy, vol. 14 (1928), p. 153.

† See J. A. Schouten, Rendiconti di Palermo, vol. 50 (1926), pp. 142–169, in particular p. 158. I do not find that this work, which is closely related to our process, gives a clear account of the fact that the coordinates ξ must be tied up with the x as described above.

For the complete development of projective infinitesimal geometry we must, in my estimation, add three independent ideas to Cartan's scheme; the first and most important of these consists in connecting the coordinates ξ with the x by the requirement of "semi-osculation," the second answers the question to what extent the geodesics determine the projective connection by the invariantive "specialization." The third idea, which I now consider, is due to T. Y. Thomas: it is possible to give the variables η themselves, and not only their ratios, a geometrical interpretation. The analytic expression in coordinates η for any projective mapping which leaves O invariant,

$$\bar{\eta}^i = \sum_k a_k{}^i \eta^k, \quad \bar{\eta}^0 = \eta^0 + \sum_k \alpha_k \eta^k, \tag{4}$$

can be so normalized that the coefficient $a_0^0 = 1$. This normalization is useful because it is not destroyed by composition: the group $\mathfrak{G}_0$ is replaced by the isomorphic group of affine transformations in $n+1$ dimensions of the form (4). If the transition $\eta \to \bar{\eta}$ on T_P corresponds to the transition $x \to \bar{x}$ on M the transformation (4) can further be described as agreeing with the transformation of the differentials $dx^0, dx^1, \cdots, dx^n$ in P when the additional coordinate x^0 transforms in accordance with the law

$$\bar{x}^0 = x^0 - \frac{1}{n+1} \log (\bar{x}x). \tag{5}$$

We now have an $(n+1)$-dimensional manifold M^* instead of M; each point of M is replaced by a filament of M^* along which $x^1, \cdots, x^n$ are constant and only x^0 varies. By (5) the distance between points on the same filament, i.e. the difference of their x^0 coordinates, as well as the filaments themselves, have an invariantive significance. An $(n+1)$-dimensional affine tangent plane, the domain of the variables η, is associated with each point of M^*. The transformation of the η, which is related to a transformation of the x on M, is the same for all points on the same filament. Extending the Γ by adding $\Gamma_{\beta 0}^{\alpha} = \delta_\beta^\alpha$, this means that the ratios $\eta^0 : \eta^1 : \cdots : \eta^n$

of a point on the tangent plane are unaltered by displacement along the filament. The n-dimensional projective displacement on M^* defined by

$$d\eta^\alpha = -d\gamma_\beta^\alpha \eta^\beta, \quad d\gamma_\beta^\alpha = \Gamma_{\beta\rho}^{\alpha} dx^\xi, \tag{6}$$

is consequently invariantively determined by the projective displacement on M.

We must next ask if this is also true for the $(n+1)$-dimensional affine displacement expressed by the same formulas; the answer is affirmative, because our normalization is so chosen that $\Gamma_{\beta\rho}^{\alpha}$ is symmetric in β and ρ. To show this, let $\overline{\Gamma}_{\beta\rho}^{\alpha}$ be the projective connection of M evaluated in a new coordinate system $\bar{x}^i$ in the manner described above, and let $\tilde{\Gamma}_{\beta\rho}^{\alpha}$ be the components of the same affine connection on M^* expressed in terms of the coordinates $\bar{x}^i$ in the manner indicated by (6) in terms of the x^i. Then the corresponding equations (6) characterize the same projective connection, that is, $\overline{\Gamma}_{\beta\rho}^{\alpha}$ and $\tilde{\Gamma}_{\beta\rho}^{\alpha}$ can differ only by a term of the form $\delta_\rho^\alpha \lambda_\rho$. Now $\overline{\Gamma}$ as well as $\tilde{\Gamma}$ must be symmetric in the two lower indices and consequently

$$\delta_\beta^\alpha \lambda_\rho = \delta_\rho^\alpha \lambda_\beta,$$

from which we obtain by the contraction $\alpha = \rho$

$$\lambda_\beta = (n+1)\lambda_\beta, \quad \lambda_\beta = 0.$$

All that we have said here can be taken over mutatis mutandis to the conformal geometry. Here the equation

$$g_{ik} dx^i dx^k = 0 \tag{7}$$

on M is fundamental. If g_{ik} be the values of the coefficients in P then the conformal geometry on the homogeneous "tangent" plane T_P is described by the equation $g_{ik} d\xi^i d\xi^k = 0$ with constant coefficients g_{ik}; the group $\mathfrak{G}$ consists of all transformations of the ξ which leave this equation invariant and consequently depends on the point P in question. The conformal displacement PP' must be a transformation which takes the equation $g_{ik} d\xi^i d\xi^k = 0$ over into $g'_{ik} d\xi'^i d\xi'^k = 0$. Consequently this does not agree literally with the scheme de-

veloped above—furthermore $\mathfrak{G}$ does not contain the affine group (only the orthogonal one). We must expressly require that the coincidence of the line elements $\overline{PP'}$ on M and $\overline{OO_1'}$ on T_P is a conformal, and not merely an affine, relation. (1) The requirement of semi-osculation is also here sufficient to tie up the conformal coordinate system ξ uniquely with the x. (2) If the manifold is special the conformal connection is uniquely determined by equation (7). (3) The transition to homogeneous coordinates η, in which the conformal representation appears as a homogeneous linear transformation, is here accomplished, following Möbius, by

$$\eta^0 : \eta^1 : \cdots : \eta^n : \eta = 1 : \xi^1 : \cdots : \xi^n : -\tfrac{1}{2} g_{ik}\xi^i\xi^k .$$

These coordinates are subject to the relation

$$g_{ik}\eta^i\eta^k + 2\eta^0\eta = 0.$$

It is convenient for the purpose of calculation to normalize the coefficients g_{ik}, only the ratios of which are given, by the condition $|g_{ik}| = 1$. This third and last step, which was carried through by Veblen in a recent paper, proceeds as before, but the result is more complicated since we have $n+2$ variables η, whereas Thomas' extension gives but $n+1$ coordinates x. Consequently we do not arrive at an affine connection on an $(n+1)$- or $(n+2)$-dimensional manifold M^*, invariantly related to the conformal connection on M.

PRINCETON UNIVERSITY

HENRY BURCHARD FINE—IN MEMORIAM

Dean Fine was one of the group of men who carried American mathematics forward from a state of approximate nullity to one verging on parity with the European nations. It already requires an effort of the imagination to realize the difficulties with which the men of his generation had to contend, the lack of encouragement, the lack of guidance, the lack of knowledge both of the problems and of the contemporary state of science, the overwhelming urge of environment in all other directions than the scientific one. But by comparing the present average state of affairs in this country with what can be seen in the most advanced parts of the world, and extrapolating backwards, we may reconstruct a picture which will help us to appreciate their qualities and achievements.

Henry Burchard Fine was born in Chambersburg, Pennsylvania, September 14th, 1858, the son of Lambert Suydam Fine and Mary Ely Burchard Fine. His father, a Presbyterian minister, died in 1869 leaving his widow with two sons and two daughters. Mrs. Fine lived with her children for a while at Ogdensburg, New York, and afterward at Winona, Minnesota, and in 1875 brought them to Princeton to complete their education. She was, by all accounts, a woman of great ability and force of character and launched all her children on honorable careers.

Thus Fine's early years were those of a country boy under pioneer conditions. He was always enthusiastic about the two great rivers, the St. Lawrence and the Mississippi, beside which he lived in those years and on which he learned to row. Rowing, by the way, was the only college sport in which he took an active part, though he was always keen on athletics and gave much time and energy to the direction of them up to the end of his life. During the years when I knew him he got much of his exercise on his bicycle, which he used both in going to and from his classes and for long rides in the country.

He finished his preparation for college during his first year in Princeton and entered the College of New Jersey, as it was then called, in 1876. Throughout his college course he was ranked at the head of his class and he graduated with first honors in 1880. As an undergraduate he specialized in the Classics, which he expected to teach later on, and he also began the study of Sanskrit. Besides this he found time for a normal undergraduate's interest in athletics, and played the flute in the college orchestra. He served for three years on the board of the college paper, the Princetonian, of which he was managing editor in his senior year. It was in this connection that he formed a life-long friendship with Woodrow Wilson.

The picture which one gets of his undergraduate years fits in completely with the impression which those of us got who knew him only in the last three decades of his life. We saw that he was widely read in literature and history, that he was fond of good music and active in bringing good concerts to Princeton, and that he took a keen interest in the games and the daily life of the undergraduates.

It seems that his adoption of mathematics as a career was largely due to George Bruce Halsted, that extraordinary enthusiast for non-euclidean geometry. Halsted had graduated from Princeton in 1875 and after studying under Sylvester at the Johns Hopkins, spent the years from 1878 to '81 as "instructor in post-graduate mathematics" in Princeton. Halsted communicated his enthusiasm to a little group of undergraduates and in particular to Fine, who had a natural bent for questions of a logical order.

On graduating, Fine was appointed Class of 1860 Fellow in Experimental Science. In this capacity he did a certain amount of experimental work and even published a paper, *On the shadows obtained in the glow discharge*, in the American Journal of Science, vol. 21 (1881) pp. 394–5, in collaboration with W. F. Magie who was subsequently Professor of Physics in Princeton. But Fine was never really interested in experimental work and in 1881 gladly accepted a position as Tutor in Mathematics, which he held until 1884.

In March 1884 he and Magie sailed together for Germany. According to his own account, he knew very little German and almost no mathematics. Nevertheless he developed so rapidly that he obtained the Ph.D. in Leipzig in May, 1885. He used to tell how Klein advised him, in spite of his ignorance, to come into the advanced lectures, and assured him that he would be learning mathematics even if he did not seem to understand everything as it went along. Fine did so and after a few months was surprised, on looking back at his early notes, to find that the whole story was clear to him. The thesis subject first proposed by Klein (on enumerative geometry) turned out to be unsuitable and, with Klein's consent, he changed to another subject suggested by Study. The latter, also a student of Klein's, has remained one of Fine's closest friends. After taking his degree, Fine spent the summer semester of 1885 in Berlin where he listened to Kronecker's lectures on the theory of eliminations and was profoundly influenced by them.

After this brief contact with the main current of mathematics, Fine returned to Princeton as an assistant professor, in which position he remained until 1889. During this period he published four papers in the American Journal of Mathematics. The first was his thesis, *On the singularities of curves of double curvature*, vol. 8 (1886), p. 156–177. The second was an extension of the results of this paper to n dimensions, vol. 9, pp. 180–184. The other two were: *On the functions defined by differential equations with an extension of the Puiseux polygon construction to these equations*, vol. 11 (1889), pp. 317–328, and *Singular solutions of ordinary differential equations*, vol. 12 (1890) pp. 295–322.

He was married September 6, 1888 to Miss Philena Fobes of Syracuse, N. Y. In 1889 he was promoted to a full professorship and in 1898 was appointed Dod professor of mathematics.

In the summer of 1891 he had another short period of study in Germany during which he became better acquainted with Kronecker. There is a reference to this in his charmingly written obituary notice of the latter, *Kronecker and his arithmetical theory of the algebraic equation*, Bulletin of the New York Mathematical Society, vol. 1 (1892), pp. 173–184. Fine remained

an admirer and student of Kronecker's work throughout his life, as is witnessed by his important paper, *An unpublished theorem of Kronecker respecting numerical equations*, this Bulletin, vol. 20 (1914), pp. 339–358. This was Fine's retiring address as President of the American Mathematical Society.

From this time onward Fine's mathematical activities were devoted more and more to teaching and to the logical exposition of elementary mathematics, though he always kept up an active interest in higher analysis, and indeed published a paper of some importance, *On Newton's method of approximation* (Proceedings of the National Academy of Sciences, vol. 2, pp. 546–552) as late as 1917.

His first book, *The Number System of Algebra treated Theoretically and Historically* (Boston and New York, Leach Shewell and Sanborn, 1891), was devoted in a single-minded way to the exposition of the logical foundations of analysis. It is still a very useful reference book on this subject. The subsequent books were all intended as text-books for undergraduates. They were *A College Algebra* (1905), *Coordinate Geometry* (with Henry Dallas Thompson) (1909) and *Calculus* (1927). Of these, the first and the last are the most successful and probably are not excelled by any other books in their respective fields from the point of view of accuracy of statement and the adequacy with which they represent the subject. They are admirably concise and free from unnecessary verbiage. In every case they had been printed or mimeographed in a preliminary form and used with classes for many years, and modified time and again as a result of this experience, before they were published.

In 1903 shortly after Woodrow Wilson became President of Princeton, Fine was made Dean of the Faculty and his energies were devoted chiefly to university administration. When Wilson resigned to become Governor of New Jersey in 1910, Fine carried the chief burden of the university administration for two years until another president was appointed. He then resigned as Dean of the Faculty but was induced to continue as Dean of the Departments of Science, which he did till his death.

His firmness of character and wisdom in counsel made him the preeminent figure among his colleagues of the Princeton faculty up to the very end. He was the man to whom they turned more than to anyone else for leadership or advice, both in public and private affairs. These qualities were widely recognized outside Princeton, for he had more than one opportunity to become a University President, and President Wilson made several efforts to induce him to take important posts in his administration. But as he said quite simply, he preferred to continue as a professor of mathematics.

In 1905, when the Princeton faculty was being enlarged because of the institution of the preceptorial system, Fine was in a position to determine the character of the new personnel in the scientific departments. He used the opportunity so well as to bring about a development in these departments which has had a vital effect on the scientific history of the whole country. To mention only a few instances in other departments than his own, which show the style of his choices, Fine was instrumental in calling or

in promoting Richardson in Physics, Russell in Astronomy, Hulitt in Chemistry, and Conklin in Biology.

In his own department, the expansion of 1905 meant the promotion of Eisenhart and Gillespie to preceptorships and the calling in of Bliss, Veblen, and Young to the same rank, and Jeans to a professorship. When Bliss and Young were called away Birkhoff and Wedderburn were called in, and later Boutroux in succession to Birkhoff. Without cataloguing more names it can be said that Fine's department has had the services of a considerable proportion of the best mathematicians of America during this period. In every case these men were called in before they became well known, at the time when this recognition meant the most to them. Though many of them stayed only a few years their contact with Fine and his group was important both to them and in the continuing growth of this group as an organism. No doubt this organism has had a greater influence on the mathematical enterprise in general because its constituents have changed rather rapidly.

In Fine's last years he was able to seize another opportunity for the advancement of science. Largely because of the confidence which he inspired, the General Education Board offered Princeton University a million dollars for research in pure science on condition that the University raise two additional millions for the same purpose. The fund was completed in 1928, chiefly by the contributions of Mr. Thomas D. Jones, a life-long friend of Fine's. Mr. Jones graduated from Princeton in 1876 before Fine entered, but they became acquainted and learned each others qualities during Wilson's administration as President of Princeton while Mr. Jones was one of the Trustees. This friendship is commemorated by the Henry B. Fine Professorship of Mathematics founded by Mr. Jones, and the Thomas D. Jones Professorship of Mathematical Physics founded by Miss Gwethalyn Jones. As a further memorial, Mr. Jones and his niece are giving a building for the department of mathematics which is to bear the name of Fine.

Fine was one of the founders of the American Mathematical Society. In January 1891, Professor T. S. Fiske as secretary of the New York Mathematical Society wrote to six prominent mathematicians asking their cooperation in enlarging the scope of this society and thus enabling it to publish a Bulletin. Of these six, Fine and Woolsey Johnson promised unqualified support. With this, and the somewhat less definite backing of the others, Fiske and his colleagues went ahead with another appeal to a wider circle of mathematicians. The first number of the Bulletin of the New York Mathematical Society appeared before the end of the same year. It records (p. 54) that Fine was appointed to fill a vacancy on the Council of the Society at the October meeting in 1891. The name of the Society was changed to the American Mathematical Society in 1894 and the first number of the present Bulletin was issued in the Autumn of that year. Fine was elected a member of the Council of the reorganized society at its first Annual Meeting in December 1894. He was President in 1911 and 1912.

In the latter part of his life Fine had to suffer the loss of two of his children and of his wife, who died in April, 1928. His own death was the

result of an accident. In the evening of December 21, 1928, he was riding his bicycle on a road in the outskirts of Princeton and was struck from behind by an automobile, the driver of which failed to see in the uncertain evening light that he was starting to make a left turn. He died early the next morning without having recovered consciousness.

The last time I saw him was in Princeton in August 1928 just after his return from a trip to Europe in which he had revisited old scenes and old friends and had recovered to some extent, in the distractions of travel, from his personal sorrows. He looked strong and well and seemed to have many years of activity before him. He spoke with humorous appreciation of the change he had observed in the attitude of European mathematicians toward their American colleagues and with pride of the esteem in which he had found his own department to be held. But most of our talk was of the future.

Shortly after this I was myself in Europe and met some of those who had seen and conversed with him on his last trip, most of them younger men who had met him for the first time. They all spoke of his charm of manner and the impression they had gained of a man of serene strength and poise and wisdom. It was a striking testimony to the way in which a man's character can be written into his appearance and manner by a long and strenuous life.

OSWALD VEBLEN

DE VRIES ON FOUR DIMENSIONS

Die vierte Dimension, eine Einführung in das Vergleichende Studium der Verschiedenen Geometrien. By Dr. H. K. de Vries. Translated into German from the second Dutch edition by Frau Dr. Ruth Struik. Wissenschaft und Hypothese, XXIX, Leipzig and Berlin, Teubner, 1926. ix+167 pp.

Although there is no mention of non-euclidean geometry in the title, nor even on the title-page, nearly half of the book is devoted to it. In fact, we have here two books put together under the title of the first. The books are independent: either could have been put first or published alone. Only in one place in the second (pages 139–141) is there introduced a little of the geometry of four dimensions, and at the very end there is an outline of the rise of this geometry, a little more than two pages, that might have been put at the end of the first part.

The treatment of the book is not strictly mathematical. We have rather a popular exposition, although considerable mathematical detail is suggested in the explanations, giving some idea of the way in which these subjects have been developed as well as the mere results of their development.

A small book, not too condensed, must leave out many things. Thus we find nothing about hyperprisms, hyperpyramids, hypercylinders, and hypercones, and nothing about hypervolume. Some might have chosen these and given only a paragraph to the regular polytopes or hypersolids. Many details of the non-euclidean geometry are presented very briefly. But the choice of material is in general excellent, and the book is well planned to impart some knowledge of these two branches of mathematics.

In both parts of the book much is said about the historical development of the subjects considered, about their usefulness, and about the frame of mind in which the reader should study them, and many interesting remarks are thrown in. Thus of the value of extending the chain of dimensions beyond the third we are told (page 22) that the notion of value is in high degree relative, and what is of inestimable value for one man leaves another entirely cold. But for the mathematician and for everyone who has an interest in scientific thought, it is of great value to know that a four or five or n dimensional geometry is possible, and the mathematician will not be satisfied to know that it is possible, but will want to know more exactly about it, as always when a new field of investigation appears.

At the beginning the relations of the simpler elements, lines, planes, and hyperplanes, their determinations and intersections, parallelism and perpendicularity, and the various kinds of angles formed by them, are gone into very thoroughly and carefully, and this study is the best sort of preparation even for a superficial understanding of the more complicated figures and relations of hyperspace.

The notion of point-value of a space is most helpful, for the dimension-number d is 1 less than w, the number of points necessary to determine a given space, and some formulas are obtained more easily by consider-

ing w. Thus two spaces, R_{d_1} and R_{d_2}, having in common a space $R_{d_{12}}$, determine a space R_d, the point-values of these spaces being connected by the relation $w=w_1+w_2-w_{12}$ and therefore their dimension-numbers by the relation $d=d_1+d_2-d_{12}$, which may also be used to determine the dimension of the intersection of two given spaces lying in and determining another given space. After this formula has been applied to spaces in R_3 and in R_4, it is suggested that the reader experiment with R_5 and R_6, and then we are told (page 35) that he will not be able to suppress the feeling of having extended very widely his horizon, and that this is the reward for the striving necessary at the beginning.

Turning to the second part, the non-euclidean geometry, we find the quadrilateral of Saccheri and his three hypotheses taken as the starting-point, with a brief indication of the results to which these three hypotheses lead. There is also an account of the way in which the hyperbolic and elliptic geometries are represented as the geometries on the pseudo-sphere and sphere, and by projection on a plane. The *boundary-curve* or *horocycle*, and the *equidistant-curve* are described (page 150) as new figures of hyperbolic geometry, and the polar relation of lines and points and of planes and points in the elliptic geometry (page 130), while the two kinds of elliptic geometry, the double and simple elliptic geometries, are mentioned (page 133). But the treatment of all of these details is very brief.

We close with an outline of the contents of this book.

After an introduction of about three pages we have Part First, euclidean geometry of several dimensions [chiefly of four dimensions], pages 4–92, and Part Second, non-euclidean geometry, pages 92–165.

The first part may be divided roughly as follows:

Points, lines, planes, and spaces, their intersections, point-values, etc., pages 4–35.

Infinitely distant elements and parallelism, pages 35–45.

Perpendicularity and rotation, the hypersphere, pages 45–59.

Angles, the number of angles between two spaces, the two angles between two planes, pages 60–77.

The regular polytopes, pages 77–92.

An outline of the second part, with some changes in order, will be:

The postulates of Euclid and the three hypotheses of Saccheri, pages 92–112.

The sum of the angles of a triangle and its area, pages 112–116, 153–161.

Euclidean geometry, pages 116–120.

Hyperbolic geometry, pages 120–125, 145–153.

Elliptic geometry, pages 125–134.

Geometrical representations, pages 134–145, 161–163.

And finally, a historical paragraph in regard to geometry of n dimensions, pages 163–165.

H. P. MANNING

PICARD ON PARTIAL DIFFERENTIAL EQUATIONS

Leçons sur quelques Types Simples d'Equations aux Dérivées Partielles avec des Applications à la Physique Mathématique. By Émile Picard. Paris, Gauthier-Villars, 1927. i+214 pp.

This book forms the first volume of a new mathematical series, *Cahiers Scientifiques*, edited by G. Julia. From the preface to volume 3 of the series, one learns with regret that Picard has abandoned the project of completing volume 4 of his *Traité d'Analyse*, which was to give the theory of partial differential equations. Instead, the subject matter of this volume will be presented in several volumes of *Cahiers Scientifiques*. The book under review contains lectures given at the Sorbonne in 1907 and repeated, with some additions, in 1925.

The first four lectures deal with the simplest and best known of the equations of parabolic type, namely Fourier's equation

$$\partial u/\partial x = \partial^2 u/\partial y^2.$$

Lecture 1 begins with the existence theorem, shows that non-analytic solutions are possible for analytic initial data, and derives Fredholm's example of a power series whose circle of convergence is a natural boundary. Lecture 2 derives the fundamental solution, shows how this is used to solve the boundary problem $u=f(x)$ for $t=0$, and as an application gives Weierstrass' original proof of his theorem on the approximation to a continuous function by polynomials. Lecture 3 applies the preceding results to various boundary problems in heat conduction, and Lecture 4 gives an account of Lord Kelvin's investigation of the propagation of an electric current along a cable.

Lectures 5 to 11 deal with various aspects of the theory of integral equations. In Lecture 5, a brief account of the Fourier integral is given and applied to an integral equation which was set up in Lecture 3. Lecture 6 deals with the properties of the integrals

$$\int_0^\infty \cos xy\phi(y)dy \text{ and } \int_0^\infty e^{-xy}\phi(y)dy$$

considered as functions of x. In Lecture 7, an integral equation of the first kind is set up for the solution of Dirichlet's problem (in two dimensions) by a potential of a single layer on the boundary curve, and the main results of Fredholm on integral equations of the second kind are enumerated.

In Lecture 8, the singular integral equations of the second kind

$$f(x) + \lambda \int_0^\infty \cos x\xi f(\xi)d\xi = \phi(x)$$

and

$$f(x) + \lambda \int_{-\infty}^\infty e^{-|x-\xi|} f(\xi)d\xi = \phi(x)$$

are briefly considered. Lecture 9 deals with the integral

$$\frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-x},$$

where $f(z)$ is defined only on the closed curve C (so that $f(z)$ is not necessarily analytic), and the principal value of the integral, when x approaches C, is defined. Lecture 10 raises the question of the necessary and sufficient condition to be imposed on $f(z)$ on C in order that the above integral shall be an analytic function (which is then $=f(x)$ by Cauchy's formula). The condition is

$$f(x) = \frac{1}{\pi i} \text{ Princ. val. } \int_C \frac{f(z)}{z-x}dz$$

for every x on C. Lecture 11 applies the principal values to the solution of Riemann's problem of finding two analytic functions one regular inside C and the other outside, such that their ratio shall take prescribed values on C. Hilbert's problem of determining an analytic $f(z)=u(x, y)+iv(x, y)$ such that $au+bv+c=0$ on C, where a, b, and c are functions of position on C, is also considered, and the lecture closes with a discussion of the tangential derivative of a single layer potential. With Lecture 12, the author returns to partial differential equations and gives a brief survey of Cauchy's problem and the theory of characteristics. Lectures 13 to 16 deal mainly with linear equations of the hyperbolic type, and the method of successive approximations is used to solve the following boundary problems: (1) the values of u are given on two characteristics; (2) u and one of its first derivatives are given along a curve which is not a characteristic (Cauchy's problem); (3) u is given on a characteristic and also on a non-characteristic; (4) u given on two non-characteristics. In Lectures 17 and 18, the problems of Lectures 13 to 16 are treated by Riemann's integration method. This method is applied to the equation of a vibrating string and to the telegraph equation in Lecture 19. Lecture 20 introduces a method due to Poincaré for the telegraph equation, and this gives occasion for some remarks on integral equations of the first kind, particularly that of Abel.

The author now passes to linear equations of the second order and elliptic type. Lecture 21 gives the proof of the analyticity of the solutions of the general second-order equation of the elliptic type, and Lecture 22 presents some results on the rather delicate question of the existence of the normal derivative of the solution. Lecture 23, which is rather loosely connected with the preceding ones, finally gives some properties of the spherical potential of Boussinesq.

Many results and still more proofs in this book are new, and the exposition has all the qualities of lucidity and elegance which one is accustomed to expect from its distinguished author.

T. H. GRONWALL

SHORTER NOTICES

Applications of the Theory of Elliptic Functions to the Theory of Numbers. By P. S. Nazimoff. Translated from the Russian by Arnold E. Ross. Mimeographed by O. E. Brown. Evanston, 1928. 146 pp.

The original of this remarkable book was presented as a magister thesis in Russia and was printed in Russian at Moscow in 1884. Abstracts of certain parts of it were published shortly after its Russian appearance, both in the Fortschritte and in a prominent French journal. Neither abstract did the book justice. That this assertion of fact is so can be seen from the following. Had an adequate account of the contents of this book appeared at the time of its publication, much later work in the theory of numbers of representations of integers in special quadratic forms need never have been published. For example, in 1914, Bulyguin, using the methods of the first few chapters of Nazimoff's book, gave a complete solution, by comparatively elementary means, of the determination of the numbers of representations of integers as sums of any even number of squares. This work itself, like the neglected Nazimoff's, seems to have escaped the notice of the professional abstractors. As late as 1919, G. Humbert, in a letter to the present reviewer, inquired what had become of this most remarkable work of Bulyguin; it was already in print. Episodes such as this but emphasize the lamentable ignorance under which most of us labor concerning Russian work in the theory of numbers. Reverting for a moment to the question of representations as a sum of an even number of squares, we may state that many recent and elaborate investigations were out of date before they were printed, on account of the work of the neglected Russian school.

In the present translation only those parts of Nazimoff's book have been presented that contain matter which was original at the time of publication and a great part of which, in the light of the above remarks, is still as fresh as it was when published. The title sufficiently describes the contents of the work. Nazimoff made the first systematic attempt to obtain everything arithmetical that is implicit in the theory of elliptic functions. That such a vast undertaking appears only rough hewn in the final product, is not to the author's discredit. Precisely where his work is least finished is the place where it is most suggestive. The problem of discovering everything that is implied for the theory of numbers by the multiply periodic functions is a well-put and solvable one; this book took the first significant step toward a solution. It is worthy of the attention of all those interested in the algebraic applications of analysis to the theory of numbers—which is quite a different thing from the analytic theory of numbers.

For those interested, the general problem which will be suggested by a reading of pages 138–144 of the present translation, is worthy of the closest attention. A complete solution of the obviously implied problem, extended to functions of any number of variables, contains the solution of any

arithmetical problem whatever that is solvable by the application of algebraic functions of elliptic theta functions.

The translation was made from what is apparently the unique copy of this book in the United States. The translators wish it to be stated that copies can be purchased only at the University of Chicago Book Store, 5802 Ellis Ave., Chicago, and that the price is $3.00. The 146 pages of mimeographing are, on the whole, clearly done; the book is substantially bound in a stiff paper cover and, as such work goes, is of a very high quality. It is to be hoped that this translation will at last make Nazimoff's penetrating ideas familiar to those who either do not read Russian or who are content to accept second hand abstracts from those who also do not read Russian.

E. T. BELL

The new Quantum Mechanics. By G. Birtwistle. Cambridge University Press, 1928. xiii+290 pp.

This book contains an exposition of the theories of Heisenberg, Schrödinger and Dirac, with applications mainly to the theory of spectra. There are also chapters on de Broglie's particle waves, and the Bose-Einstein and Fermi-Dirac statistics. The closing chapter deals with Heisenberg's indetermination relations and the formulation given to them by Bohr in his article in Nature, April 1928 (and communicated to the author before publication; the author's preface is dated Copenhagen, 1 Oct. 1927).

The book contains a great deal of information on the main developments in quantum theory up to the latter part of 1927, and is convenient for reference; on the other hand, owing perhaps to lack of time, the presentation is uncritical, so that the book cannot replace the study of the original memoirs.

T. H. GRONWALL

Traité d'Analyse, volume 3, third edition. By Émile Picard. Paris, Gauthier-Villars, 1928. ix+660 pp.

This is a reprint of the second edition, with additional notes on pp. 601–656. Although no review of previous editions has appeared in the Bulletin, the *Traité* is so well known and so highly appreciated by the mathematical public, that it seems superfluous to describe the contents of the main body of the volume. The additions to the new edition are as follows: (1) a note on Sundman's work on the problem of three bodies (reprinted from Bulletin des Sciences Mathématiques, 1913, pp. 313–320); (2) a lecture on integral invariants and Poisson stability (from the author's course at the Sorbonne in 1914, printed here for the first time); (3) remarks on some of Poincaré's results in analytical mechanics (Bulletin des Sciences Mathématiques, 1914, pp. 320–236); (4) on the solution of $\Delta u = e^u$ on a closed Riemann surface (Journal für Mathematik, vol. 150); (5) on linear partial differential equations and the generalization of Dirichlet's problem (proof of the analyticity of the solution for a second-order equation of the elliptic type, reprinted from Acta Mathematica, vol. 25 (1901), pp. 121–137).

T. H. GRONWALL

Vorlesungen über Darstellende Geometrie. By Dr. Emil Müller. II. Band: *Die Zyklographie.* Edited from the manuscript by Dr. Josef Leopold Krames. Leipzig and Vienna, Franz Deuticke, 1929. ix+476 pp.

The third edition of the second volume of Dr. Müller's text and the first volume of his lectures were reviewed by the present reviewer in this Bulletin in 1923 (vol. 29, pp. 478–479). The first volume of the lectures was in the name of Dr. Müller and Dr. Erwin Kruppa.

Dr. Krames states in the introduction that the late Dr. Müller entrusted to him the editing of the manuscript for this second volume, only part of which was then in form for publication.

This book contains a more extensive use of the orientation of circles than had previously been made by Fiedler. In fact, the authors' fundamental operations are with orientated circles, (Zykeln), orientated straight lines (Speeren), and orientated curves (Richtungskurven). Their use of these terms is in harmony with that of Ed. Laguerre.

The ten chapters deal with the "zyklographic" representation of points, straight lines, planes, curves, and nets of curves, and surfaces and with contact transformations, etc. Special attention is directed to the chapter on "*C*-Geometrie" (parabolic pseudo geometry).

We are endebted to Dr. Krames for his skill in editing this unfinished manuscript—the last contribution of a talented and industrious worker in the field of descriptive geometry.

E. B. COWLEY

Repertorium der Höheren Mathematik. By E. Pascal. Second edition: first volume, *Analysis*; second section. Edited by E. Salkowski. Leipzig, B. G. Teubner, 1927. Pp. v–xii and 529–1023.

Parts of the second edition of the Pascal Repertorium have been reviewed, upon their appearance, in this Bulletin, and the outstanding features of the work have been fully discussed. It is therefore unnecessary to make very detailed comments on the present section. Reference is made to the Bulletin: vol. 19, pp. 372–374, and vol. 29, p. 373.

Approximately two-thirds of this second section of the first volume is devoted to the theory of analytic functions in its classical ramifications, and a good exposition is given of both the Weierstrassian and Riemannian points of view. A relatively large portion of the space thus reserved for function theory is assigned to the automorphic functions and the elliptic modular functions. The remainder of the book offers a brief treatment of fundamental concepts and methods in (1) the theory of differential and difference equations, and that of differential forms, (2) the theory of continuous transformation groups, and of contact transformations, and (3) the calculus of variations. Contributors to the section are: A. Guldberg of Oslo, Ernesto Pascal of Naples, F. Engel of Giessen, Hans Hahn of Vienna, Gustav Doetsch of Stuttgart, E. Jahnke and A. Barneck of Berlin, H. W. E. Jung of Halle, and R. Fricke of Braunschweig.

In conformity with the general plan of the undertaking, the present section, like its predecessors in the second edition, attains a definitely

higher degree of readability than one has hitherto been accustomed to expect in professedly encyclopedic summaries of mathematical science. Although proofs are, of necessity, frequently omitted, theorems are stated with clearness and accuracy, and in well-arranged sequence. References to the literature, while by no means exhaustive, will in most cases provide ample orientation for those who wish to undertake specialized studies.

By reason of the arrangement of the material and the style of exposition, it is probable that most mathematicians will find in this section of the Repertorium not only appreciable stimulation but also a measure of downright enjoyment.

L. S. HILL

Tables of Damped Vibrations. By W. E. Milne. University of Oregon Publication. Mathematics Series. Vol. 1, No. 1, March 1929. 39 pp.

In an earlier paper entitled *Damped vibrations. General theory together with solutions of important special cases*, dated August, 1923, the author gave fairly extensive tables covering the case in which the damping is proportional to the square of the velocity, and short tables applicable to the more general situation in which the resistance function is of the form $R=v(B\pm Dv)$, where v is the velocity and B and D are positive constants. The sign to be taken is the same as that of v.

The objects of the present paper are (a) to furnish more extensive tables for the solution of the problem of oscillations with a resistance function of the type cited and (b) to supply additional tables applicable to the case where the term involving v^2 does not change sign when the sense of the motion reverses. A concrete application of case (b) is afforded by the oscillation of water in the hydraulic surge chamber.

The fourteen tables,—all of four decimal places,—are preceded by useful preparatory paragraphs on the "Scope of Application, The Hydraulic Surge Chamber, Interpolation and Method of Computation." In the first of these attention is called to the fact that, in many cases, the binomial resistance function quoted above may represent experimental data as satisfactorily as the conventional relation $R=Cv^m$. In particular, when m lies between 1 and 2, (as is often the case in practice), it is shown that it is possible to determine B and D in such a manner as to make the error incurred by using the easily calculable binomial in place of the mth power monomial resistance formula of no greater importance than the other uncertainties inherent in the problem. For illustration, the "standard error" attains its greatest magnitude of about 0.0082 when 1.4 is the value of m. Since the tables furnish solutions of differential equations which cannot be solved in terms of known functions they will be welcomed by mathematicians, physicists, and engineers when problems of damped oscillations arise in their investigations and demand numerical solution. The computation of the tables was aided materially by a liberal grant of funds from the National Academy of Sciences.

H. S. UHLER

Exercices d'Analyse, vol. 1. By Gaston Julia (edited by René Harmegnies and Roger Julia). Paris, Gauthier-Villars, 1928. vii+454 pp.

This book is a worthy descendent of a long line of French *Exercices sur le calcul infinitésimal*. Such collections of problems are intended primarily for the students who prepare themselves for the licence or the agrégation and contain problems of the type set in these examinations. A thorough knowledge of the theory is expected as well as skill in calculation and the training is directed towards developing both qualities in the students.

The present book contains a small number of carefully chosen problems, each problem followed by one or more complete solutions. About two-thirds of the first volume is devoted to the applications of analysis to geometry. An admirable account of the theory of Fourier series (pp. 120–190) is eminently suitable as outside reading for first year graduate students. This part of the book will probably be found the most useful one to the general mathematical public outside of France. The printing is good, but when will MM. Gauthier-Villars return to the use of good paper in their books?

EINAR HILLE

A History of Physics. Revised and enlarged edition. By Florian Cajori. New York, Macmillan, 1929.

The first edition of this book gave in 285 pages a concise yet detailed account of the important phases of the development of physics from ancient times to the time of publication in 1899. While many additions and alterations have been made, the new edition is substantially the same as the old so far as this part of the work is concerned. The important change in the second edition is due mainly to the excellent summary, covering about 100 pages, of the development of physics during the first twenty-eight years of the twentieth century.

The method of presentation adopted in the first edition, that is, the arrangement of the material by periods, and within a period according to the well known sub-divisions of physics, has been retained. In this connection it is interesting to note the space allotted to different periods. Thus ancient civilizations are given 22 pages; the Middle Ages and the Renaissance 32 pages; the seventeenth and eighteenth centuries 83 pages; the nineteenth century 147 pages; and the first quarter of the twentieth century, 116 pages. Taking this distribution as a fair measure of the progress of physics during these periods and taking into account the length of each period, we find that the increase of our knowledge in physics has followed practically an exponential curve, indicating that our knowledge of the physical world is not only not tending to a saturation value, but its rate of increase is rising with time.

One of the commendable features of the book consists in the numerous references to original sources.

Professor Cajori has earned the gratitude of scientists by his works on the history of science.

H. M. DADOURIAN

NOTES

By invitation of the Program Committees, several addresses will be delivered at meetings of the Society during the Fall, as follows:

At New York City, October 26, 1929, by Professor Tomlinson Fort, on *The general theory of factorial series.*

At Ann Arbor, November 29, 1929, by Professor Tomlinson Fort, on *The general theory of factorial series;* and by Professor L. M. Graves, on *Discontinuous solutions in the calculus of variations.*

At Pasadena, November 29, 1929, by Professor M. W. Haskell, on *Autopolar configurations in two and in three dimensions.*

At the Annual Meeting of the Society, to be held at Lehigh University, Bethlehem, Pennsylvania, on December 27–28, 1929, both the morning and the afternoon of Saturday will be devoted to a symposium on the general topic, *The differential equations of engineering.* By invitation of the Program Committee, addresses will be given by Professor H. W. March, The University of Wisconsin, on *The problem of diffusion;* by Professor Vannevar Bush, The Massachusetts Institute of Technology, on *Integraphs for differential equations;* by Dr. T. H. Gronwall, Columbia University, on *Numerical integration of differential equations;* by Dr. Ing. A. Nadai, The Westinghouse Electric and Manufacturing Company (formerly Professor of Applied Mathematics, University of Goettingen), on *Plasticity and related problems of non-rigid bodies;* by Mr. R. H. Park, Engineering Department, The General Electric Company, on *Analytic determination of flux plots;* by Professor S. Timoshenko, The University of Michigan (formerly with the Westinghouse Company), on *Problems in elasticity.* Professor Tomlinson Fort is chairman of the Committee on Arrangements.

At the meeting of the Society at Des Moines, Iowa, in conjunction with the meetings of the American Association for the Advancement of Science and of the Mathematical Association of America, on December 30–31, 1929, Professor L. L. Dines will give an address on *Linear inequalities and some related properties of functions* before the joint meeting of the Society and the Mathematical Association; and Professor O. D. Kellogg will give an address on *An unsolved problem of uniqueness in potential theory.* The annual Josiah Willard Gibbs address by Professor Irving Fisher on *Applications of mathematics to the social sciences,* will be held on Tuesday, December 31. A special announcement concerning this meeting will be issued.

At the meeting in New York City, February 21, 1930, by invitation of the Program Committee, Mr. E. C. Molina will give an address on *The theory of probabilities.*

The Colloquium Editorial Committee wishes to call attention to the fact that in addition to the Colloquium Lectures delivered before the Society, it is in a position to print a few other volumes. Authors are invited

to submit manuscripts. For further information, see page 100 of volume 32 of this Bulletin.

A new volume of the Colloquium Series, Coble's *Algebraic Geometry and Theta Functions*, has appeared and may be obtained through the New York Office of the Society. The edition of Veblen's *Analysis Situs* (Cambridge Colloquium, Part II) being exhausted, a new edition will be prepared and will probably appear in 1930.

The tercentenary of the birth of Christiaan Huygens was celebrated at Leiden in April, 1929, under the auspices of the Royal Academy of Sciences and the Senate of the University of Leiden.

On June 29, 1929, the Berlin Physical Society celebrated the fortieth anniversary of Professor Max Planck's entrance into scientific work. That Society has created a medal in his honor, to be awarded for distinguished work in mathematical physics (see this Bulletin, vol. 34, p. 792). On this occasion Professor Planck presented the first cast of this medal to Professor Albert Einstein, emphasizing particularly his influence on the development of the quantum theory.

The Alfred Ackermann-Teubner memorial prize for 1928 has been awarded to Professor Albert Defant, director of the Institut für Meereskunde, Berlin, for his work collected and published in *Das Zeitenproblem des Meeres in Landesnähe*, Hamburg, 1925.

The Adams prize of Cambridge University for 1927–28 has been awarded to Professor Sydney Chapman, of the Imperial College of Science Technology, London, for a memoir on the announced subject *The variations in the earth's magnetic field in relation to electric phenomena in the upper atmosphere and on the earth.* The Sudbury Hardyman prize of Emmanuel College, Cambridge, has been awarded to A. H. Wilson, for a dissertation on *Quantum mechanics.*

The medal of the class of 1889, Columbia University, has been awarded to Dr. Irving Langmuir, of the General Electric Company, for "distinguished achievement in improving the incandescent electric light."

Professor Frank Schlesinger, of Yale University, has been awarded the Bruce medal of the Astronomical Society of the Pacific.

On the occasion of the Abel centennary celebration, the University of Oslo conferred honorary degrees on the following mathematicians, in addition to those listed in the last issue of this Bulletin (pp. 583–584): Professors L. E. J. Brouwer of Amsterdam, Friedrich Engel of Giessen, Rudolph Fueter of Zürich, Kurt Hensel of Marburg, Christian Juel of Copenhagen, Edmund Landau of Göttingen, E. L. Lindelöf of Helsingfors, L. E. Phragmén of Stockholm, Salvatore Pincherle of Bologna, Teiji Takagi of Tokyo, Charles de la Vallée-Poussin of Louvain, Oswald Veblen of Princeton, and Hermann Weyl of Zürich.

Professor Jules Drach, of the Sorbonne, has been elected a member of the Paris Academy of Sciences in the section of mechanics, as successor to the late Professor J. Boussinesq.

The Royal Academy dei Lincei has elected Professor G. Peano (Turin) national member; Professors G. D. Birkhoff (Cambridge, Mass.), C. Carathéodory (Munich), A. Sommerfeld (Munich) foreign associates.

The Accademia Pontificia in Rome has elected Father G. Boccardi (retired professor of the University of Turin) and Professor E. Pistolesi, of the University of Pisa, ordinary members, and Professor T. Levi-Civita, of the University of Rome, corresponding member.

Associate Professors L. Brusotti and F. Cecioni, of the University of Pisa, G. Giorgi, of the University of Cagliari, G. Marletta, of the University of Catania, and E. G. Togliatti, of the University of Genoa, have been promoted to professorships.

The University of Paris has conferred an honorary doctorate on Professor Albert Einstein.

Professor Leon Lichtenstein, of the University of Leipzig, has been elected a member of the Polish Academy of Sciences, Cracow.

Professor I. Schur, of the University of Berlin, has been elected a corresponding member of the Leningrad Academy of Sciences.

The University of Glasgow has conferred an honorary doctorate on Professor H. S. Carslaw, of the University of Sydney and Emmanuel College, Cambridge.

Cambridge University has conferred honorary doctorates on Professor Paul Langevin and Sir Frank Dyson.

Dr. Irving Langmuir, of the General Electric Company, has received an honorary doctorate from the Charlottenburg Technical School.

Among the recent elections to the National Academy of Sciences are the following: Dr. C. J. Davisson, of the Bell Telephone Laboratories; Professor Bergen Davis, of Columbia University; Professor William Hovgaard, of the Massachusetts Institute of Technology.

The following have been elected to the American Academy of Arts and Sciences: fellows, Professors Marston Morse and J. L. Walsh, of Harvard University; foreign honorary members, Professors V. F. K. Bjerknes, of the University of Oslo, Ludwig Prandtl, of the University of Göttingen, Willem de Sitter, of the University of Leiden, and Hermann Weyl, of the Zurich Technical School.

Honorary doctorates have been conferred on Professor A. H. Compton, of the University of Chicago, by Yale University, and by Ohio State University.

The University of Michigan has conferred the honorary degree of doctor of laws on Professor R. A. Millikan, of the California Institute of Technology.

Columbia University has conferred an honorary degree on Dr. G. A. Plimpton, trustee of Barnard College, known to mathematicians for his collections of mediaeval mathematical works.

Harvard University has conferred an honorary doctorate on Professor H. N. Russell, of Princeton.

Swarthmore College has conferred an honorary doctorate on Dr. W. F. G. Swann, director of the Bartol Foundation of the Franklin Institute.

Guggenheim fellowships for 1929 have been awarded to the following persons: Dr. Dinsmore Alter, professor of astronomy, University of Kansas, for research on the application of statistical methods to meteorological and astronomical problems; Dr. O. S. Duffenbach, assistant professor of physics, University of Michigan, for an investigation of molecular impact phenomena in gases and the relation of these to the excited states of molecules and the chemical binding of their atoms; Dr. Olive C. Hazlett, assistant professor of mathematics, University of Illinois, for the continuation of studies of the arithmetics of linear associative algebras; Dr. R. J. Kennedy, research associate, California Institute of Technology, for the continuation of researches in the theory of radiation; Dr. W. M. Latimer, associate professor of chemistry, University of California, for the study of the thermodynamic treatment of solutions with a view to extending work on the entropy of aqueous ions; Dr. R. S. Mulliken, associate professor of physics, University of Chicago, for work on theoretical problems connected with the subject of the formation and dissociation of molecules and the assignment of quantum numbers for electrons in molecules, based on the study of band spectra; Dr. J. C. Slater, assistant professor of physics, Harvard University, for research in quantum mechanics; Dr. C. H. Smiley, instructor in mathematics, University of Illinois, to investigate methods of determining orbits of comets and asteroids; Dr. J. H. Van Vleck, professor of theoretical physics, University of Wisconsin, for research in quantum mechanics; Dr. G. T. Whyburn, adjunct professor of mathematics, University of Texas, for research in point set theory, with particular emphasis on the structure of continua and of continuous curves. The above list includes renewals.

Professor A. Fraenkel, of the University of Kiel, has declined a permanent professorship at the University of Jerusalem, but has accepted an appointment for the academic years 1929–30, 1930–31. Dr. R. Schmidt will take his place at the University of Kiel during this interval.

Dr. Vojtěch Iarník, of the Czech University at Prague, has been promoted to an associate professorship of mathematics.

Dr. Heinrich Ott has been promoted to an associate professorship of theoretical physics at the University of Würzburg.

Associate Professor Bernhard Schilling, of the Dresden Technical School, has been appointed professor of mathematics at the University of Santiago, Chile.

Associate Professor Víktor Trkal, of the Czech University at Prague has been promoted to a professorship of theoretical physics.

Dr. W. Hurewicz has been admitted as privat docent at the University of Amsterdam.

Professor J. Sun, of Yenching University, Pekin, has been appointed head of the department of astronomy at North-Eastern University, Mukden, China.

At Cambridge University, F. C. Burkill, of Peterhouse, P. A. M. Dirac, of St. John's, and F. P. Ramsey, of King's, have been appointed university lecturers in mathematics. Dr. W. M. Smart, of Trinity College, chief assistant at the Observatory, has been appointed to the John Couch Adams astronomership.

Mr. E. C. Titchmarsh, of the University of London, has been appointed professor of pure mathematics at the University of Liverpool.

Associate Professor B. A. Bernstein, of the University of California, has been promoted to a professorship of mathematics.

Professor Florian Cajori, of the University of California, has retired, as professor emeritus of the history of mathematics.

Associate Professor C. C. Camp, of the University of Nebraska, has been promoted to a professorship.

Assistant Professor E. F. A. Carey, of the University of Montana, has been promoted to an associate professorship.

Assistant Professor E. U. Condon, of Princeton, has been appointed professor of theoretical physics at the University of Minnesota.

Dr. C. M. Cramlet, National Research Fellow, has been appointed assistant professor of mathematics at the University of Washington.

Associate Professor J. C. Fitterer, of the Colorado School of Mines, has been promoted to a professorship.

Dr. William Gafafer has been appointed research associate at Johns Hopkins University.

Mr. W. H. Gage, of Victoria College, British Columbia, has been promoted to an assistant professorship of mathematics.

Associate Professor C. A. Garabedian, of the University of Cincinnati, has been appointed associate professor of mathematics at St. Stephen's College, Columbia University.

Professor W. V. N. Garretson, of Ouachita College, Arkadelphia, has been appointed associate professor of mathematics at the Oklahoma Agricultural and Mechanical College.

Assistant Professor H. C. Hicks, of the University of Oregon, has been appointed professor of mathematics and aeronautics at Texas Technological College.

Assistant Professor J. W. Hurst, of Montana State College, has been promoted to an associate professorship.

Dr. C. G. Jaeger, of the University of Missouri, has been appointed to an assistant professorship at Tulane University.

Assistant Professor R. A. Johnson, of Hunter College, has been promoted to an associate professorship.

Miss Elizabeth Knight, of the Milwaukee State Normal School, has been promoted to an assistant professorship.

Miss Edna Kramer has been appointed assistant professor of mathematics at the New Jersey State Teachers College, Upper Montclair.

Assistant Professor C. H. Langford, of the University of Washington, has been appointed associate professor of philosophy at the University of Michigan.

Mr. T. R. Long has been promoted to an assistant professorship at the University of Rochester.

Associate Professor C. E. Love has been promoted to a professorship at the University of Michigan.

Mr. C. I. Lubin, of the University of Cincinnati, has been promoted to an assistant professorship.

Mr. C. T. Male, of Union College, has been promoted to an assistant professorship.

Mr. A. E. Meder has been promoted to an assistant professorship at the New Jersey College for Women, Rutgers University.

Dr. T. W. Moore, National Research Fellow, has been appointed research associate at Princeton University.

Dr. G. W. Myers, professor of the teaching of mathematics and astronomy in the school of education at the University of Chicago, has retired.

Mr. G. A. Parkinson, of the University of Wisconsin, has been promoted to an assistant professorship.

Mr. C. S. Porter, of Amherst College, has been promoted to an associate professorship.

Associate Professor B. P. Reinsch, of Southern Methodist University, has been promoted to a professorship of mathematics.

Assistant Professor H. P. Robertson, of the California Institute of Technology, has been appointed assistant professor of mathematical physics at Princeton University.

Dr. I. M. Sheffer, National Research Fellow, has been appointed assistant professor of mathematics at Pennsylvania State College.

Assistant Professor J. H. Taylor, of the University of Wisconsin, has been appointed professor of mathematics at George Washington University.

Mr. G. C. Vedova has been promoted to an assistant professorship at St. John's College, Annapolis.

Assistant Professor R. L. Wilder, of the University of Michigan, has been promoted to an associate professorship.

Mr. F. G. Williams, of Cornell University, has been appointed professor of mathematics at Susquehanna University.

The following appointments to instructorships are announced:

Columbia University, Dr. A. A. Albert.

College of the City of Detroit, Dr. K. W. Folley, Dr. D. C. Morrow.

Harvard University, Mr. A. E. Andersen, Mr. A. E. Currier, Mr. G. A. Hedlund, Mr. S. H. Kimball, Mr. G. B. Morrey, Mr. S. B. Myers, Mr. G. Sauté, Mr. G. B. Van Schaack, Mr. C. Wexler.

Long Island University, Mr. E. A. Knobelauch.

Purdue University, Mr. E. L. Klinger, Mr. J. M. Thompson.

Rice Institute, Dr. L. M. Blumenthal.

Worcester Polytechnic Institute, Mr. S. A. Lepeshkin.

Professor Henri Andoyer, of the Sorbonne, member of the Paris Academy of Sciences in the section of astronomy, died June 12, 1929, at the age of sixty-six.

Dr. Karl Heun, professor of theoretical mechanics at the Karlsruhe Technical School, died January 10, 1929.

Professor E. Hoppe, of Göttingen, known for his work in the history of mathematics, died August 12, 1928.

George Birtwistle, fellow of Pembroke College, Cambridge, died May 19, 1929, at the age of fifty-two.

Sir George Knibbs, of Melbourne, Australia, formerly director of the Commonwealth Institute of Science and Industry, and author of *The Mathematical Theory of Population*, died March 30, 1929, at the age of seventy.

NEW PUBLICATIONS

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BOREL (E.). Leçons sur les fonctions de variables réelles et les développements en séries de polynomes. 2e édition. Paris, Gauthier-Villars, 1928. 12+158 pp.

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CASSINA (U.). Calcolo numerico con numerosi esempi e note storiche originali. Bologna, Zanichelli, 1928. 19+453 pp.

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FORSYTH (A. R.). Mathematics in life and thought. Cardiff, University of Wales Press Board, 1929. 34 pp.

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GORDIN (J.). Untersuchungen zur Theorie des unendlichen Urteils. Berlin, Akademie-Verlag, 1929. 6+168 pp.

GRAU (J. G.). Grundriss der Logik. 3te durchgesehene und erweiterte Auflage. Leipzig, Teubner, 1929. 6+162 pp.

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HJELMSLEV (J.). See MOHR (G.).

JAVELOT (—.). Exercices d'algèbre. Paris, Vuibert, 1929. 200 pp.

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ferences. 4th edition, rewritten and enlarged. London, Macmillan, 1928. 24+548 pp.

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LATTA (R.) and MACBEATH (A.). The elements of logic. London, Macmillan, 1929. 8+393 pp.

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LORIA (G.). Histoire des sciences mathématiques dans l'antiquité hellénique. Paris, Gauthier-Villars, 1929. 215 pp.

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Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 26, 1929.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than October 5. An invited address will be given by Professor Tomlinson Fort; see page 740.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 29, 1929.

Abstracts must be in the hands of Associate Secretary B. A. Bernstein, University of California, Berkeley, Calif., not later than November 1. By invitation of the Program Committee. An invited address will be given by Professor M. W. Haskell; see page 740.

ANN ARBOR, MICHIGAN, November 29-30, 1929.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 7. Invited addresses will be given by Professors Tomlinson Fort and L. M. Graves; see page 740.

BETHLEHEM, PENNSYLVANIA, ANNUAL MEETING, December 27-28, 1929.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than November 30. A symposium on the differential equations of engineering has been arranged; invited addresses will be given by Professors H. W. March, Vannevar Bush, S. Timoshenko, Dr. T. H. Gronwall, Dr. A. Nadai, and Mr. R. H. Park; see page 740.

DES MOINES, IOWA, December 30-31, 1929.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 30. There will be the usual joint session with Section A of the American Association for the Advancement of Science and with the Mathematical Association of America. Addresses will be given by Professors L. L. Dines and O. D. Kellogg; see page 740. The Josiah Willard Gibbs address by Professor Fisher on *Application of mathematics to the social sciences,* will be held on Tuesday, December 31. More detailed announcements will be made later.

NEW YORK CITY, February 21, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than February 1. An invited address will be given by Mr. E. C. Molina on *The theory of probabilities.*

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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